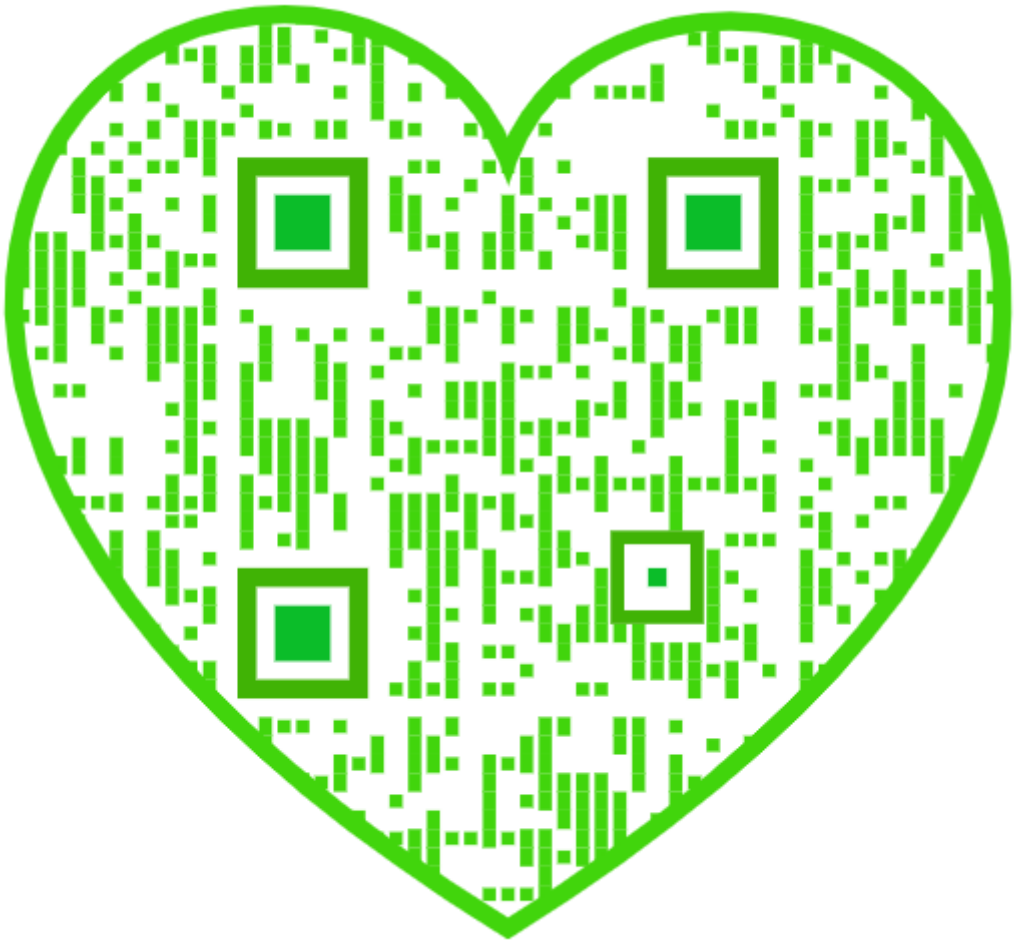


ELEMENTARY TREATISE

SPHERICAL TRIGONOMETRY

N. F. DUFUIS J. MATHESON

KINGSTON
R. UGLOW & COMPANY

The EDITH *and* LORNE PIERCE
COLLECTION *of* CANADIANA

Queen's University at Kingston

QUEEN'S
UNIVERSITY
LIBRARY

Presented by

...nonymous

KINGSTON, ONTARIO
CANADA

I $\cos a = \cos b \cos c + \sin b \sin c \cos A$ } To get 3rd side
Given two sides and included angle

$\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$ } Given 3 sides to get the angles

II $\cos A = -\cos B \cos C + \sin B \sin C \cos a$
Given two angles and the side between them to find the 3rd angle.

III $\cos a = \dfrac{\cos A + \cos B \cos C}{\sin B \sin C}$ { Given 3 angles to find the sides

IV $\sin A = \dfrac{2}{\sin b \sin c} \sqrt{\{\sin s \sin(s-a) \sin(s-b) \sin(s-c)\}}$

V $\sin A = \dfrac{1}{\sin b \sin c} \sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}$

VII $\dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c}$ = Sine Formula

VIII (a) Given 2 sides & included angle to find a second angle
(b) " 2 angles & " side " " another side

$\cot a \sin b = \cos b \cos C + \sin C \cot A$
$\cot a \sin c = \cos c \cos B + \sin B \cot A$
$\cot b \sin c = \cos c \cos A + \sin A \cot B$
$\cot b \sin a = \cos a \cos C + \sin C \cot B$
$\cot c \sin a = \cos a \cos B + \sin B \cot C$
$\cot c \sin b = \cos b \cos A + \sin A \cot C$

Not to be used for an angle under 10° ←

" " " " " ←

Not " " " " " near 180° ←

$a = 180° - A$

$2S = A + B + C$

IX $\cos\frac{A}{2} = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}$ } For finding an angle when sides are given Use Log's.

X $\sin\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}$

XI $\cos\frac{a}{2} = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B \sin C}}$ — Polar of IX

XII $\sin\frac{a}{2} \sqrt{\frac{-\cos S \cos(S-A)}{\sin B \sin C}}$ — Polar of X

$\frac{\text{Area of a Lune}}{\text{Surface of a Sphere}} = \frac{A}{2\pi}$

$\therefore$ Area of Lune $= \frac{A}{2\pi} \cdot 4\pi r^2 = \underline{2Ar^2}$

XII Speech is comic
Caprice is captious
Sums are so much
Crumbs are spare bits

1. $\sin\frac{1}{2}(A+B)\cos\frac{1}{2}C = \cos\frac{1}{2}(a-b)\cos\frac{1}{2}C$
$\cos\frac{1}{2}(A+B)\cos\frac{1}{2}C = \cos\frac{1}{2}(a+b)\sin\frac{1}{2}C$
$\sin\frac{1}{2}(A-B)\sin\frac{1}{2}C = \sin\frac{1}{2}(a-b)\cos\frac{1}{2}C$
$\cos\frac{1}{2}(A-B)\sin\frac{1}{2}C = \sin\frac{1}{2}(a+b)\sin\frac{1}{2}C$

13. $\sin \frac{E}{2} = \frac{\sin A \sin b \sin C}{4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}} = \frac{\sin B \sin C \sin a}{4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{a}{2}} = \frac{\sin C \sin a \sin b}{4 \cos \frac{a}{2} \cos \frac{b}{2} \cos \frac{c}{2}}$

14. $\tan \frac{E}{4} = \sqrt{\tan \frac{S}{2} \tan \frac{S-a}{2} \tan \frac{S-b}{2} \tan \frac{S-c}{2}}$

E = Sph. excess $= A + B + C - \pi$

Geo. E. Bolton
Science 12

SPHERICAL TRIGONOMETRY

AN

ELEMENTARY TREATISE

ON

SPHERICAL TRIGONOMETRY

WITH APPLICATIONS TO

GEODESY AND ASTRONOMY

INTENDED FOR THE USE OF

PRACTICAL SCIENCE STUDENTS

BY N. F. DUPUIS AND J. MATHESON

KINGSTON, ONT.
R. UGLOW & COMPANY
1907

CA 535 I 02 1907 Cop 2

Entered according to the Act of the Parliament of Canada, in the year one thousand nine hundred and seven, by R. Uglow & Company, Kingston, Ont., at the Department of Agriculture.

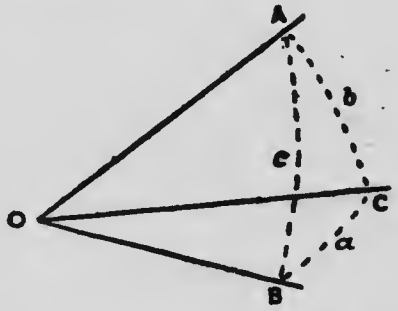
PRINTED AT THE JACKSON PRESS
KINGSTON, ONT.

Elements of Spherical Trigonometry.

1. As the triangle is the fundamental figure in plane Trigonometry, so the *trihedral angle* or *3-faced corner* is the fundamental figure in spherical or spatial Trigonometry.

A 3-faced corner consists essentially of six parts, namely, 3 face angles and 3 dihedral angles.

Thus, in the figure, the three non-complanar lines OA, OB, OC form the edges of the 3-faced corner.



The angles AOB, BOC, and COA are the face-angles of the corner; and the angles between the pairs of planes determined by AOB and AOC, BOC and BOA, COA and COB, are the dihedral angles of the corner.

If a sphere with centre at O cuts the edges in A, B, C, the figure of section made by the spherical surface consists of a triangle of which the sides are arcs of great circles, namely, AB, BC, and CA, and the angles of the triangle are the dihedral angles of the three-faced corner. This formation of a triangle on the surface of a sphere gave rise to the name Spherical Trigonometry.

But it should be constantly kept in mind by the student, that what is called a spheric triangle, and to

405802

which spherical Trigonometry directly applies, however it may be graphically represented, has no necessary relation to a sphere. The spheric triangle is virtually a 3-faced corner of which the face angles are called the sides of the triangle, and the dihedral angles are called the angles of the triangle.

And thus a spheric triangle, unlike a plane one, has no linear extension; and all its six parts, whether called sides or angles, are in reality angles. So that spheric Trigonometry is properly the trigonometry of space; or it may be called tridimensional trigonometry, or the trigonometry of direction only.

2. From the nature of the spheric triangle as set forth in the preceding article it follows that the properties of a 3-faced corner are also properties of the spheric triangle, although these properties are generally set forth in different terms in the two cases. The prominent properties of the 3-faced corner are given in the author's synthetic solid geometry under the pages referred to in the following parallelisms:

3-faced Corner.	Spheric Triangle.
(i)	(i)
Any two face angles are together greater than the third. *Page 32.*	Any two sides are together greater than the third.
(ii)	(ii)
The greater dihedral angle is opposite the greater face angle; and	The greater angle is opposite the greater side; and conversely, the great-

conversely, the greater face angle is opposite the greater dihedral angle.—*Page 36.*

(iii)

When two dihedral angles are equal the opposite face-angles are equal; and conversely, when two face angles are equal the opposite dihedral angles are equal.—*Page 37.*

(iv)

The sum of the face-angles is less than four right angles.—*Page 39.*

(v)

The sum of the dihedral angles lies between 2 right angles and 6 right angles.—*Page 41.*

er side is opposite the greater angle.

(iii)

When two angles are equal the opposite sides are equal; and conversely, when two sides are equal the opposite angles are equal.

(iv)

The sum of the sides is less than four right angles.

(v)

The sum of the angles of a spheric triangle lies between 2 right angles and 6 right angles.

Thus the sum of the angles of a spheric triangle, unlike that for the plane triangle, is not a fixed quantity. The amount by which the sum of the angles exceeds two right angles, or

$A+B+C-\pi$, is the

spherical excess.

(vi)

When the three dihedral angles of a 3-faced corner are given the face angles are given also. —*Page 41.*

(vi)

When the angles are given the sides are given also, and thus unlike the plane triangle, a spheric triangle is determined by its three angles.

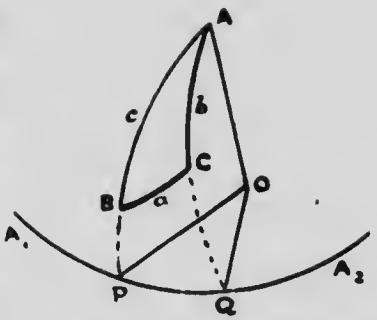
3. The Polar triangle.—This triangle plays so important a part in the theory of the spheric triangle that its nature and its relation with the primitive triangle should be carefully mastered.

It is shown in Solid Geometry (page 40) that to every three-faced corner there corresponds another corner called its reciprocal, and such that the face angles of the one are respectively the supplements of the dihedral angles of the other, and that this property is reciprocal.

So, interpreting this with respect to the spheric triangle, we say that to every spheric triangle there corresponds a second triangle such that the sides of the one are respectively the supplements of the angles of the other, this property being reciprocal. One of these triangles is said to be the **polar** of the other ; or they are polars of each other.

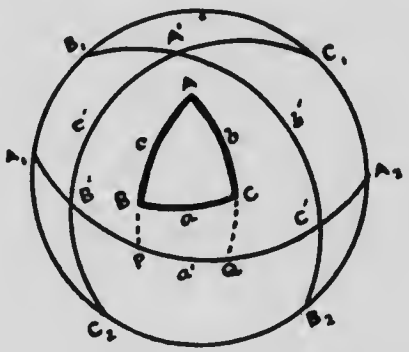
4. The relation between a spheric triangle and its polar may also be considered as follows:—

It is usual to represent a spheric triangle by three segments of great circles on a sphere of indefinite radius, as AB, BC, CA, or c, a, b in the figure, these segments denoting the sides of the triangle.



Let O be the centre of the sphere. From A as centre and with the arc A P = one fourth of the great circle, or $\angle AOP = 90°$, describe the circle A_1PQA .....This is a *great circle of the sphere*, as its plane passes through the centre *O*. This circle is the *equator* to the point A, and A is a *pole* of the equator A_1PQA_2...., the point diametrically opposite A being the other pole. Produce AB and AC to meet the equator in P and Q respectively. Draw OA, OP, OQ.

The angle *A* is the angle between the planes APO and AQO. But as AOP and AOQ are right angles, POQ measures this angle. So that the angle subtended at the centre by P Q is the angle *A*.



Now let A B C be a triangle on the surface of a sphere, and let A_1A_2, B_1B_2, C_1C_2, be respectively the equators of A, B, and C.

These great circles determine the triangle A′B′C′ which is polar to ABC.

As A′B = 90° and A′C = 90°, the equator to A′ passes through B and C, so that A′ is a pole of BC. Similarly B′ is a pole of CA, and C′ of AB; and the triangle ABC is thus polar to A′B′C′, and hence the triangles are *polar to each other*.

Again, $B'Q = PC' = 90°$, and $PQ = \angle A$, the radius of the sphere being taken as the unit-length, and it is readily seen that $B'C' = B'Q + PC' - PQ$; or $a' = 180° - A$. So that we have in general

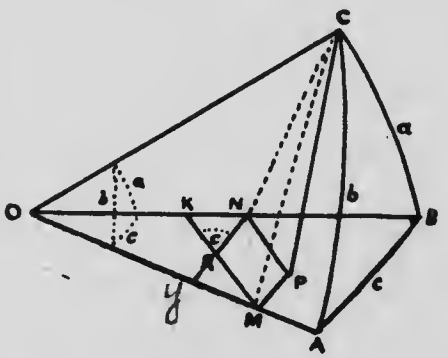
$$\left.\begin{array}{l} A + a' = B + b' = C + c' \\ = a + A' = b + B' = c + C' \end{array}\right\} = 180° = \pi \ldots . (43)$$

And thus when we know the sides of a triangle we know the angles of its polar, and when we know the angles of a triangle we know the sides of its polar.

5. The cosine Formula.—As in plane Trigonometry, so in spheric, there are certain formulae, expressing relations between sides and angles, which must be considered as being fundamental. The first of these, called the cosine formula, we proceed to develop.

A B C is a spheric triangle of which the sides are $a = \angle BOC$, $b = \angle COA$, $c = \angle AOB$; and of which the angles are, A = angle between planes AOB and AOC, B = angle between planes BOC and BOA, and C = angle between the planes C O A and C O B.

C P is perpendicular on the plane A O B; P M is perpendicular on OA, PN is perpendicular on OB, MK is parallel to PN, and NR is parallel to PM,



Then, NR = PM, and CM is ⊥ to OA,

and CN is ⊥ to OB, and $\angle NRK = \angle BOA = c$.

Now
$$\cos c = \frac{OK}{OM} = \frac{ON}{OM} - \frac{KN}{OM}$$
$$= \frac{ON}{OC} \cdot \frac{OC}{OM} - \frac{KN}{NR} \cdot \frac{PM}{CM} \cdot \frac{CM}{OM}$$

But
$$\frac{ON}{OC} = \cos a, \quad \frac{OC}{OM} = \frac{1}{\cos b}, \quad \frac{KN}{NR} = \sin c, \quad \frac{PM}{CM} = \cos A,$$

and $\frac{CM}{OM} = \tan b.$

Hence by substitution

$$\cos c = \frac{\cos a}{\cos b} - \sin c \,.\, \cos A \,.\, \tan b.$$

And multiplying by cos b and transposing we get

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

This is a fundamental form., and by varying the letters in cylic order we get the set—

$$\left.\begin{array}{l} \text{(i)} \ \cos a = \cos b \cos c + \sin b \sin c \cos A \\ \text{(ii)} \ \cos b = \cos c \cos a + \sin c \sin a \cos B \\ \text{(iii)} \ \cos c = \cos a \cos b + \sin a \sin b \cos C \end{array}\right\} \ldots\ldots (44)$$

And from these by transposition—

$$\left.\begin{array}{l} \text{(i)} \ \cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c} \\ \text{(ii)} \ \cos B = \dfrac{\cos b - \cos c \cos a}{\sin c \sin a} \\ \text{(iii)} \ \cos C = \dfrac{\cos c - \cos a \cos b}{\sin a \sin b} \end{array}\right\} \ldots\ldots (45)$$

The three formulas in each set are practically one and the same, as it is immaterial how the letters a, b, c are disposed about the sides of the triangle.

The formula (44) serves to find the third side when two sides and the included angle are given; and the formula (45) to find the angles when the three sides are given.

These formulae are not logarithmic, and if used in this form natural functions must be employed.

Ex. 1. Given $b = 47^\circ 20'$, $c = 56^\circ 10'$, $A = 44^\circ 10'$, to find the side a.

Employing (i) set (44)—

cos $47^\circ 20'$	$= 0{\cdot}67773$		sin $47^\circ 20'$	$= 0{\cdot}73531$	
cos $56^\circ 10'$	$= 87655^*$		sin $56^\circ 10'$	$= 66038^*$	
	33886			58825	
	3389			2206	
	407			44	
	47			4	
	5			·61079	(2)
add	·37734	(1)	cos $44^\circ 10'$	23717^*	
	·43812			42755	
cos a	$= {\cdot}81546$			611	
a	$= 35^\circ 22'$			427	
				18	
				1	
				·43812	(3)

*This being the multiplier the order of digits is reversed in order to make use of contracted multiplication, and the star shows the position of the units place in the multiplier.

(1) This is the product cos b cos c; (2) This is the product sin b sin c; and (3) is the product sin b sin c cos A.

Ex. 2. Given $a=35^\circ 22'$, $b=47^\circ 20'$, $c=56^\circ 10'$, to find A. Employing (i) of set (45)—

cos b = 0·67773	sin b = 0·73531
cos c = 87655*	sin c = 66038*
Product ·37734	Product ·61079
cos a = ·81546	
Difference = ·43812	

Then ·43812 ÷ ·61079 = ·71732 = cos A,

and $A=44^\circ 10'$.

Ex. 3. Find the two remaining angles, B and C, from the data of Ex. 2.

The sum of the three angles thus found will be greater than two right angles, and the excess over two right angles is the *spherical excess* for that particular triangle.

Ex. 4. Find the spherical excess for the triangle of Ex. 2.

EXERCISE.

1. Given $a=50^\circ 45'$, $b=78^\circ 20'$, $C=60^\circ$, to find c.

2. Given $b=62^\circ 50'$, $c=80^\circ 5'$, $A=134^\circ$, to find a.

3. In an equilateral spheric triangle if a denotes a side and A an angle, $\cos A\ (1+\cos a)=\cos a$.

4. Find the spherical excess for the equilateral triangle whose side is $44^\circ 42'$.

5. If each side of a triangle be 90°, show that the spherical excess is 90°.

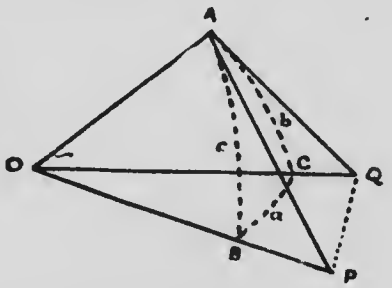
6. If each side of a triangle be 60°, then the spherical excess is $\pi - 3\cos^{-1}\frac{1}{3}$

7. If r be the earth's radius, and the side of a triangle on it be a°, show that the length of the side is $\pi r a/180$.

8. Find the length of 1° on the earth, its radius being 3960 miles.

9. A ship starting from the equator sails a straight course for 306 miles. She then changes her course through 123°, and by taking a straight course for 280 miles reached the equator again. Find the distance in miles between the point of departure and the point of arrival.

10. ABC is a spheric triangle and AP and AQ are tangents at A, to the sides c and b respectively.



Then

$$PQ^2 = AP^2 + AQ^2 - 2AP\cdot AQ\cos A\ ;$$

Also $PQ^2 = OP^2 + OQ^2 - 2OP\cdot OQ\cos a.$

Subtracting and dividing throughout by OA^2,

$$1 = \frac{OP}{OA} \cdot \frac{OQ}{OA} \cos a - \frac{AP}{OA} \cdot \frac{AQ}{OA} \cos A.$$

But $\frac{OP}{OA} = \sec c$, $\frac{OQ}{OA} = \sec b$, $\frac{AP}{OA} \tan c$, $\frac{AQ}{OA} = \tan b$.

By substitution and reduction obtain the cosine formula.

6. **The polar cosine formula.** The formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

expresses a relation between the sides a, b, c and the angle A in any triangle. But from (43) if A', B', C', a', b', c' be the parts of the polar triangle, $a = \pi - A'$, $b = \pi - B'$, $c = \pi - C'$ and $A = \pi - a'$,

And substituting these in the cosine formula—

$$\cos (\pi - A') = \cos (\pi - B') \cos (\pi - C') + \sin (\pi - B') \sin (\pi - C') \cos (\pi - a')$$

expresses a relation between the angles A′, B′, C′ and the side a' of the polar triangle. But every triangle is a polar to some triangle. Therefore, dropping accents, and remembering that $\cos (\pi - \theta) = -\cos \theta$, and $\sin (\pi - \theta) = \sin \theta$, we get, by changing signs,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

This is the polar of the cosine formula, and as before we have the two sets.

$$\left.\begin{array}{ll} \text{(i)} & \cos A = -\cos B \cos C + \sin B \sin C \cos a. \\ \text{(ii)} & \cos B = -\cos C \cos A + \sin C \sin A \cos b. \\ \text{(iii)} & \cos C = -\cos A \cos B + \sin A \sin B \cos c. \end{array}\right\} \ldots (46)$$

and

$$\left.\begin{array}{ll} \text{(i)} & \cos a = \dfrac{\cos A + \cos B \cos C}{\sin B \sin C} \\ \text{(ii)} & \cos b = \dfrac{\cos B + \cos C \cos A}{\sin C \sin A} \\ \text{(iii)} & \cos c = \dfrac{\cos C + \cos A \cos B}{\sin A \sin B} \end{array}\right\} \ldots\ldots (47)$$

Form (46) determines the third angle when two angles and the side between them are given ; and form (47) finds a side when the three angles are given.

It will be noticed that both (44) and (46) are cosine formulae, and in deriving the one from the other sides have been changed to angles and angles to sides, with necessary changes of algebraic signs.

Exercise.

1. Given $A = 125°$, $B = 73°20'$, $c = 20°47'$, to find angle C.

2. A and B are on the equator and 720 miles apart. A great circle line is run from A in a north-east course, and another from B in a north-west course. Find the angle at which the courses meet, and the distance of the point of meeting from A.

3. Given the angles of a triangle as 65°, 85° and 55°, to find the sides.

7. First transformation of Cosine Formula.

We have

$$\sin^2 A = 1 - \cos^2 A = 1 - \left\{\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right\}^2 \ldots (2)$$

This expression can be reduced in two different ways, giving rise to two apparently different results.

1st. Factoring (2) as the difference of two squares,

$$\sin^2 A = \left\{1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c}\right\} \left\{1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c}\right\}$$

$$= \frac{\cos a - \cos (b+c)}{\sin b \sin c} \cdot \frac{\cos (b-c) - \cos a}{\sin b \sin c}$$

$$= \frac{4 \sin \tfrac{1}{2}(a+b+c) \sin \tfrac{1}{2}(b+c-a) \sin \tfrac{1}{2}(a+c-b) \sin \tfrac{1}{2}(a+b-c)}{\sin^2 b \sin^2 c}$$

and denoting $a+b+c$ by $2s$, and therefore $a+b-c$ by $2(s-c)$ &c., this reduces to

$$\sin A = \frac{2}{\sin b \sin c} \sqrt{\{\sin s \sin (s-a) \sin (s-b) \sin (s-c)\}} \ldots (48)$$

2nd. Reducing (2) by squaring out the bracket

$$\sin^2 A = \frac{\sin^2 b \sin^2 c - \cos^2 a - \cos^2 b \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c}$$

But $\sin^2 b \sin^2 c = (1 - \cos^2 b)(1 - \cos^2 c)$

$= 1 - \cos^2 b - \cos^2 c + \cos^2 b \cos^2 c$,

and this substitution gives

$$\sin A = \frac{1}{\sin b \sin c} \sqrt{\{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c\}} \ldots (49)$$

Formulae (48) and (49) express the sine of an angle in terms of the three sides, and are sometimes useful. Form (48), consisting of factors only, is logarithmic, or adapted to the use of log-functions. But (49) can be used only with natural functions.

8. It will be noticed from both (48) and (49) that the product sin A sin b sin c is given by an expression symmetrical in all the sides, so we must have

sin A sin b sin c = sin B sin c sin a = sin C sin a sin b,

and dividing throughout by sin a sin b sin c we get

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \quad \ldots\ldots\ldots\ldots\ldots\ldots (50)$$

and this relation is called the **sine formula.**

The sine formula may be proved independently from the figure of art. 5 as follows :

$$\frac{\sin A}{\sin a} = \frac{CP}{CM} \cdot \frac{CO}{CN} = \frac{CP}{CN} \cdot \frac{CO}{CM} = \frac{\sin B}{\sin b}.$$

and then from symmetry each equals $\dfrac{\sin C}{\sin c}$

Consider any two angles and the sides opposite them, as A, B, a, b. When any three are given the sine formula enables us to find the fourth.

Ex. 1. Given $a = 48°45'$, $b = 37°13'$ and $A = 58°10'$, to find B.

Here $$\sin B = \frac{\sin A \cdot \sin b}{\sin a}$$

Now sin A = 0·84959, sin b = 0·60483

sin a = 0·75184.

$$\therefore \sin B = \frac{0{\cdot}84959 \times 0{\cdot}60483}{0{\cdot}75184} = 0.68346 = \sin B$$

Hence $B = 43^\circ 7'$ or $136^\circ 53'$,

and from (ii) of art. 2, as $a > b$ so $A > B$

$$\therefore B = 43^\circ 7'$$

Ex. 2. If $a = 48^\circ 45'$, $b = 37^\circ 13'$ and $B = 43^\circ 7'$, we readily find that $A = 58^\circ 10'$ or $121^\circ 50'$, and as $a > b$ the solution is ambiguous.

9. From (48) and (49) it is seen that the expressions

$$2\sqrt{\{\sin s \sin (s-a) \sin (s-b) \sin (s-c)\}}$$

and $\sqrt{\{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c\}}$ are identical in value, the first being the factored form of the second. Denote either by u.

Then $\quad u = \sin A \sin b \sin c.$

To interpret u geometrically.

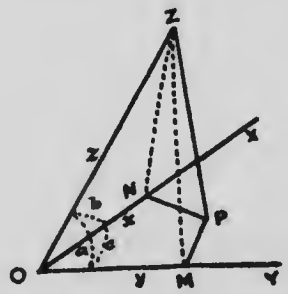
O—XYZ is a three faced corner, ZP is normal to the plane XOY, and PN is ⊥ to OX,

Then $\angle ZNP = A$, and

$$\sin A = \frac{ZP}{ZN} = \frac{ZP}{z \sin b}$$

$$\therefore ZP = z \sin A \sin b$$

Also, $xy \sin c$ = the area of the parallelogram on OX, OY as sides



$$\therefore \quad u = \sin A \sin b \sin c = \frac{ZP \times \text{plgm. } OX \cdot OY}{xyz};$$

$$\text{or,} \quad u = \frac{\text{Vol. of ppd. on } xyz}{\text{Vol. of cuboid on } xyz} \quad \ldots\ldots\ldots\ldots \quad (51)$$

Now the volume of the cuboid on xyz is xyz,

$$\therefore \quad \text{Vol. of ppd on } xyz = xyzu \ldots \quad \ldots\ldots\ldots (52)$$

Ex. 1. To find the volume of a parallelepiped, the direction edges being 3, 4, 5 and the face angles 30°, 60°, 75°.

(i) Taking natural functions,

$\cos 30° = 0{\cdot}86603$, $\cos 60° = 0{\cdot}50000$, $\cos 75° = 0{\cdot}25882$

$$\cos^2 30° + \cos^2 60° + \cos^2 75° = 1{\cdot}06700$$

$$1 + 2 \cos 30° \cos 60° \cos 75° = 1{\cdot}22414$$

$$\therefore \quad u = \sqrt{\{1{\cdot}22414 - 1{\cdot}06700\}} = 0{\cdot}3964$$

$$\therefore \quad \text{vol.} = 3 \times 4 \times 5 \times 0{\cdot}3964 = 23{\cdot}784.$$

(ii) Taking log-functions,

$s = 82°30'$, $s - a = 52°30'$, $s - b = 22°30'$, $s - c = 7°30'$

$$\text{l. sin } s + \text{l. sin } (s-a) + \text{l. sin } (s-b) + \text{l. sin } (s-c) = 8{\cdot}59428$$

Then $\log u = \log 2 + \frac{1}{2}(18{\cdot}59428) = 9{\cdot}59817$

$$\therefore \quad \log \text{vol.} = \log 3 + \log 4 + \log 5 + 9{\cdot}59817 = 1{\cdot}37632$$

$$\therefore \quad \text{vol.} = 23{\cdot}79.$$

EXERCISE.

1. The face-angles of one corner of a ppd. are 68°, 72°, and 84°, and the edges are 10, 12 and 15. To find its volume.

2. A cube is transformed so that each of the face angles at one corner is 60°. If V be the volume of the original and V' of the transformed, show that $2V' = V\sqrt{2}$.

3. If in Ex. 2 each face angle at one corner is 45°, then $V'\sqrt{2} = V\sqrt{(\sqrt{2}-1)}$.

4. Show that the sine formula is its own polar.

5. Find a dihedral angle of a regular tetrahedron.

6. A rhomboid has its three dihedral angles, 100°, 115°, and 120°. Find the face angles at a corner.

7. If $2s$ denote the sum of the sides of a triangle and $2S$ denote the sum of the angles of its polar, then $S+s=\frac{3}{2}\pi$.

8. Prove that

$$\sin a \sin B \sin C$$
$$= \sqrt{\{1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C\}}$$

9. Find the form polar to (48).

10. Second transformation of Cosine Formula.

(i) $\cos a = \cos b \cos c + \sin b \sin c \cos A$

(ii) $\cos c = \cos b \cos a + \sin b \sin a \cos C$

(iii) $\sin c = \dfrac{\sin a}{\sin A}\sin C$

From (i) eliminate $\cos c$ by means of (ii), and $\sin c$ by (iii), thus

$$\cos a = \cos b\,(\cos b \cos a + \sin b \sin a \cos C) + \sin b \cos A \frac{\sin a}{\sin A} \sin C;$$

or $\cos a \sin^2 b = \sin a \sin b\,(\cos b \cos C + \sin C \cot A)$.

and dividing throughout by $\sin a \sin b$ we get the first of the following set—

$$\left.\begin{array}{ll} \text{(i)} & \cot a \sin b = \cos b \cos C + \sin C \cot A \\ \text{(ii)} & \cot a \sin c = \cos c \cos B + \sin B \cot A \\ \text{(iii)} & \cot b \sin c = \cos c \cos A + \sin A \cot B \\ \text{(iv)} & \cot b \sin a = \cos a \cos C + \sin C \cot B \\ \text{(v)} & \cot c \sin a = \cos a \cos B + \sin B \cot C \\ \text{(vi)} & \cot c \sin b = \cos b \cos A + \sin A \cot C \end{array}\right\} \ldots\ldots (53)$$

This is the **cotangent formula**. Of course only one of these is necessary, and from (i) all the others are derived by a cyclic interchange of letters.

The cotangent formula solves the problems—

1st. Given two sides and the included angle, to find a second angle, and

2nd. Given two angles and the included side, to find another side.

Ex. 1. Given $a = 73°20'$, $b = 62°10'$, and $C = 124°$,

Then $$\cot A = \frac{\cot a \sin b - \cos b \cos C}{\sin C}$$

$$= \frac{0{\cdot}29938 \times 0{\cdot}88431 + 0{\cdot}46690 \times 0{\cdot}55919}{0{\cdot}82904}$$

$$= 0{\cdot}63426, \text{ and } \therefore A = 57°37'$$

Similarly

$$\cot B = \frac{\cot b \sin a - \cos a \cos C}{\sin C};$$

which gives $B = 51°13'$

The sum of the angles is thus $232°50'$ and the spherical excess is $52°50'$.

EXERCISE.

1. Given $A = 120°$, $B = 76°30'$, $c = 42°25'$, to find the remaining parts of the triangle.

2. In a 3-faced corner two of the face angles are 36° and 45° respectively, and the dihedral angle between these faces is 84°. To find the other dihedral angles.

3. The latitude and longitude of A are 32°N. and 75° W., and of B they are 55° N. and 87°40′W. Find the direction from A to B, and also from B to A, both in reference to the meridian of the place. Also find the distance from A to B in miles, the earth's radius being 3960 miles.

4. Departing from a point D on the equator we start in a north-east direction and follow a great circle for 500 miles, arriving at A. What angle at A does our course make with the meridian?

11. The three fundamental forms which have now been developed, namely—

(i) $\cos a = \cos b \cos c + \sin b \sin c \cos A$,

(ii) $\sin a : \sin A = \sin b : \sin B = \sin c : \sin C$,

(iii) $\cot a \sin b = \cos b \cos C + \sin C \cot A$,

with their polars, are sufficient to solve all cases of spheric triangles, provided we are willing to employ natural functions; for only the sine formula is logarithmic.

But there are other forms derived from these which are frequently more convenient, and which are essential if log-functions are to be employed.

Some of these we proceed to develop.

12. Half Angles.

$$2\cos^2\frac{A}{2} = 1+\cos A = 1+\frac{\cos a-\cos b\cos c}{\sin b\sin c}, \quad \text{(i, 45)}$$

$$=\frac{\cos a-\cos(b+c)}{\sin b\sin c} = \frac{2\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(b+c-a)}{\sin b\sin c};$$

and with $2s=a+b+c$ this becomes

$$+\cos\frac{A}{2}=\sqrt{\left\{\frac{\sin s\sin(s-a)}{\sin b\sin c}\right\}} \quad \ldots\ldots\ldots\ldots \quad (54)$$

In a similar manner from the relation

$$2\sin^2\frac{A}{2}=1-\cos A \text{ we get}$$

$$+\sin\frac{A}{2}=\sqrt{\left\{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}\right\}} \quad \ldots\ldots\ldots \quad (55)$$

These forms are concise and convenient for finding an angle when the sides are given, and they are both adapted to the use of log-functions.

As the sine of an angle changes most rapidly when the angle is small, and the cosine most rapidly when the angle is nearly a right angle, form (54) should not be

taken when the angle is known to be small, say under 10°, and from (55) should not be employed when the angle is nearly 180°. With these restrictions either form may be employed.

13. Polar forms, half sides.

The formulae polar to (54) and (55) may be obtained from these by the ordinary method of obtaining polar forms; or by reducing them directly from the polar of the cosine formula.

Thus, writing $180° - A$ for a, $180° - B$ for b, $180° - C$ for c, and putting $2S = A + B + C$, we have

$$2s = a + b + c = 540° - 2S.$$

$$\therefore \; s = 270° - S \; ; \; s - a = 90° - (S - A)$$

$\therefore \sin s = -\cos S$, and $\sin (s - a) = \cos (S - A)$, &c.,

whence
$$\sin \frac{a}{2} = \sqrt{\left\{ \frac{-\cos S \cos (S - A)}{\sin B \sin C} \right\}} \quad \ldots\ldots (56)$$

And in a similar way

$$\cos \frac{a}{2} = \sqrt{\left\{ \frac{\cos (S - B) \cos (S - C)}{\sin B \sin C} \right\}} \quad \ldots\ldots (57)$$

These forms find a side where the three angles are given. They may also be developed as follows:

$$2 \sin^2 \frac{a}{2} = 1 - \cos a.$$

But from (i, 47)
$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}$$

whence

$$2\sin^2\frac{a}{2} = -\frac{\cos(B+C)+\cos A}{\sin B \sin C}$$

$$= -\frac{2\cos\frac{1}{2}(A+B+C)\cos\frac{1}{2}(B+C-A)}{\sin B \sin C};$$

and finally

$$\sin\frac{a}{2} = \sqrt{\left\{\frac{-\cos S\cos(S-A)}{\sin B \sin C}\right\}}.$$

and similarly for $\cos\frac{a}{2}$.

14. Delambre's, or Gauss' formulae.

$$\sin\frac{A+B}{2} = \sin\frac{A}{2}\cos\frac{B}{2} + \sin\frac{B}{2}\cos\frac{A}{2}$$

$$= \sqrt{\left\{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}\right\}} \cdot \sqrt{\left\{\frac{\sin s \sin(s-b)}{\sin a \sin c}\right\}}$$

$$+ \sqrt{\left\{\frac{\sin s \sin(s-a)}{\sin b \sin c}\right\}} \cdot \sqrt{\left\{\frac{\sin(s-a)\sin(s-c)}{\sin a \sin c}\right\}}$$

$$= \frac{\sin(s-b)+\sin(s-a)}{\sin c}\left\{\frac{\sin s \sin(s-c)}{\sin a \sin b}\right\}^{\frac{1}{2}}$$

$$= \frac{2\sin\frac{1}{2}(2s-a-b)\cos\frac{1}{2}(a-b)}{2\sin\frac{1}{2}c\cos\frac{1}{2}c}\cos\frac{C}{2}$$

$$= \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}\cos\frac{C}{2}$$

$$\therefore \sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c = \cos\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C;$$

and this is the first of the set of four. The complete set follows—

$$\left.\begin{aligned}
&\text{(i)}\ \sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c=\cos\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C.\\
&\text{(ii)}\ \cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}c=\cos\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C.\\
&\text{(iii)}\ \sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c=\sin\tfrac{1}{2}(a-b)\cos\tfrac{1}{2}C.\\
&\text{(iv)}\ \cos\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}c=\sin\tfrac{1}{2}(a+b)\sin\tfrac{1}{2}C.
\end{aligned}\right\}(58)$$

The difficulty of remembering these formulae when wanted suggests the helpfulness of a mnemonic key.

Remembering the order of letters A, B, c—a, b, C, and that the first two are always connected by + or −, also that only half angles occur, we may use the following key as a help to remembering the proper functions.

Let s denote sine, c cosine, p plus and m minus. Then the following sentences contain the proper letters.

Speech is comic, or sine, plus, cosine—cosine, minus, cosine, which agrees with the first ; and thus all—

(i) Speech is comic.
(ii) Caprice is captious,
(iii) Sums are so much,
(iv) Crumbs are spare bits.

15. Napier's Analogies.

These are four formulae discovered by Baron Napier, and from having been written in the form of a proportion they are called *analogies*. They may be established independently, but they are best derived from Delambre's formulae, from which they come directly by division as follows : For the first divide (i) of (58) by (ii) ; for the second divide (iii) by (iv) ; for the third divide (iv) by (ii) ; and for the fourth divide (iii) by (i).

Results

$$\left.\begin{aligned}
&\text{(i)}\ \tan\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C\\
&\text{(ii)}\ \tan\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\operatorname{cct}\tfrac{1}{2}C\\
&\text{(iii)}\ \tan\tfrac{1}{2}(a+b)=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c\\
&\text{(iv)}\ \tan\tfrac{1}{2}(a-b)=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c
\end{aligned}\right\}\quad\ldots(59)$$

Forms (iii) and (iv) are polars of (i) and (ii) respectively.

(i) and (ii) taken together find the two remaining angles when the two sides and the included angle are given.

And (iii) and (iv) taken together find the two remaining sides when two angles and the side between them are given.

16. Solution of the spheric triangle.

The spheric triangle offers 6 cases for solution, these being grouped into three pairs, such that the two forming any pair are polar to one another.

They are as follows :

Case I. Given the three sides, to find an angle.
" II. Given the three angles, to find a side.
" III. Given two sides and the included angle.
" IV. Given two angles and the included side.

Case V. Given two sides and an angle opposite one of these sides.

" VI. Given two angles and a side opposite one of these angles.

17. Case I. Three sides given, to find an angle.

(*a*) With natural functions this is solved by (i, 45)

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Ex. Given $a = 82°50'$, $b = 75°\ 25'$, $c = 47°20'$,

Then taking the proper functions from the table,

$$\cos A = \frac{0{\cdot}12476 - 0{\cdot}25179 \times 0{\cdot}67773}{0{\cdot}96778 \times 0{\cdot}73531} = -0{\cdot}06447.$$

whence $A = 93°42'$.

(*b*). With log-functions we may employ form (54) or (55) put in logarithmic form, namely—

$$\left.\begin{aligned} \text{l. cos } \frac{A}{2} &= \tfrac{1}{2}\left\{\text{l. sin } s + \text{l. sin } (s-a) - \text{l. sin } b - \text{l. sin } c\right\} \\ \text{Or l. sin } \frac{A}{2} &= \tfrac{1}{2}\left\{\text{l. sin } (s-b) + \text{l. sin } (s-c) - \text{l. sin } b - \text{l. sin } c\right\} \end{aligned}\right\} (60)$$

Ex. Taking the data of the last Ex. $s = 102°48'$

$s - a = 19°58'$

l. sin s	$= 9{\cdot}989071$
l. sin $(s-a)$	$= 9{\cdot}533357$
col. sin b	$= 0{\cdot}014222$
col. sin c	$= 0{\cdot}133530$
Divide by 2	$9{\cdot}670180$
l. cos $\frac{A}{2}$	$= 9.835090$

$$\frac{A}{2} = 46^\circ 51' \qquad \therefore\ A = 93^\circ 42'$$

(*c*) Of course either of the formulae (54) or (55) may be worked with natural functions; and also we can make (47) logarithmic by means of an auxiliary angle, as follows:

Since cos *b* and cos *c* cannot be greater than 1, and are usually less, we may put cos *b* cos *c* = cos θ.

Then

$$\cos A = \frac{\cos a - \cos\theta}{\sin b \sin c} = \frac{2 \sin \tfrac{1}{2}(\theta + a) \sin \tfrac{1}{2}(\theta - a)}{\sin b \sin c}$$

Hence $\quad \text{l. cos } \theta = \text{l. cos } b + \text{l. cos } c$

and $\quad \text{l. cos } A = \text{l. } 2 + \text{l. sin } \frac{\theta + a}{2} + \text{l. sin } \frac{\theta - a}{2} - \text{l. sin } b - \text{l. sin } c$ $\Bigg\}$ ..(61)

and these are logarithmic.

Taking the foregoing example—

l. cos b = 9·401035
l. cos c = 9·831058

sum = 9·232093 = l. cos θ

$\therefore\ \theta = 80^\circ 10'$
$a = 82^\circ 50'$
$\theta + a = 163^\circ$
$\theta - a = -2^\circ 40'$

l. 2 = 0·301030
l. sin $\frac{\theta + a}{2}$ = 9·995203
l. sin $\frac{\theta - a}{2}$ = 8·366777 n
col. sin b = 0·014222
col. sin c = 0·133530

l. cos A = 8·810762 n
$\therefore\ A = 93^\circ 42'$

n after the logarithm denotes that it is the logarithm of a negative number.

18. Case II. Three angles given, to find a side.

This case may be called the polar of Case I, and the forms are polar to those of Case I.

Thus we have

(*a*) For natural functions

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}.$$

(*b*) For log-functions

$$\sin \frac{a}{2} = \left\{ \frac{-\cos S \cos (S-A)}{\sin B \sin C} \right\}^{\frac{1}{2}}, \quad (58)$$

$$\text{or } \cos \frac{a}{2} = \left\{ \frac{\cos (S-B) \cos (S-C)}{\sin B \sin C} \right\}^{\frac{1}{2}}, \quad (59)$$

or,

$$\left.\begin{aligned} &\text{l. } \sin \frac{a}{2} = \tfrac{1}{2}\{\text{l. } \cos S + \text{l. } \cos (S-A) - \\ &\text{l. } \sin B - \text{l. } \sin C\} \\ &\text{l. } \cos \frac{a}{2} = \tfrac{1}{2}\{\text{l. } \cos (S-B) + \text{l. } \cos (S-C) - \\ &\text{l. } \sin B - \text{l. } \sin C\} \end{aligned}\right\} \quad \ldots (62)$$

Also by putting $\cos B \cos C = \cos \phi$, under (*a*) we reduce the expression for cos *a* to

$$\cos a = \frac{2 \cos \tfrac{1}{2}(\phi + A) \cos \tfrac{1}{2}(\phi - A)}{\sin B \sin C}$$

$$\left.\begin{aligned} &\text{or} \qquad \text{l. } \cos \phi = \text{l. } \cos B + \text{l. } \cos C \\ &\text{and} \\ &\text{l. } \cos a = \text{l. } 2 + \text{l. } \cos \frac{\phi + A}{2} + \text{l. } \cos \frac{\phi - A}{2} \\ &- \text{l. } \sin B - \text{l. } \sin C \end{aligned}\right\} \quad \ldots . (63)$$

EXERCISE.

1. If each of the sides of a triangle be $\frac{1}{2}$ show that each of the angles is $\frac{1}{2}\pi$.

2. The north polar distance of a town A is 67°10′, and of B, 54°40′, and the distance between the places is 1348 miles. Find their difference in longitude.

3. The dihedral angles of a corner are each 120°. Find the value of a face angle.

4. A triangular hopper is to be made to have its dihedral angles each 90°. What angle must be given to the sides of the hopper?

5. An upright square pyramid has one side of its base b, and one of its edges e. Show that the secant of one of the dihedral angles is $1-4\frac{e^2}{b^2}$.

19. Case III. Two sides and the included angle given.

This case is divisible into two sub-cases according as to whether we wish to find the third side, or the two remaining angles.

(i) *The third side being required.*

The cosine formula

$$\cos a = \cos b \cos c - \sin b \sin c \cos A,$$

gives the solution, as b and c are two sides and A is the included angle. This form is of course suited to natural functions only.

This may be adapted to log-functions as follows:

$\cos b \cos c + \sin b \sin c \cos A = \cos b \sin c (\cot c + \tan b \cos A)$.

Now as a tangent or cotangent may have any value whatever, put $\tan b \cos A = \cot \theta$.

Then $$\cos a = \frac{\cos b \sin (\theta + c)}{\sin \theta}$$

or $$\left.\begin{aligned} &\text{l. cot } \theta = \text{l. tan } b + \text{l. cos } A \\ &\text{l. cos } a = \text{l. cos } b + \text{l. sin } (\theta + c) - \text{l. sin } \theta \end{aligned}\right\} \quad \ldots\ldots (64)$$

Ex. Given $b = 74^\circ 10'$, $c = 58^\circ 40'$, $A = 84^\circ$, to find a.

l. tan $b = 0{\cdot}54729$
l. cos $A = 9{\cdot}01923$
sum $= 9{\cdot}56652 =$ l. cot θ
$\therefore \quad \theta = 69^\circ 46'$
$\theta + c = 128^\circ 26'$

l. cos $b = 9{\cdot}43591$
l. sin $(\theta + c) = 9{\cdot}89395$
col. sin $\theta = 0{\cdot}02766$

l. cos $a =$ sum $= 9{\cdot}35752$

$\therefore \quad a = 76^\circ 50'$.

(ii) *The two remaining angles being required.*

This is done by formula (59).

$$\tan \tfrac{1}{2}(B + C) = \frac{\cos \frac{1}{2}(b - c)}{\cos \frac{1}{2}(b + c)} \cot \tfrac{1}{2} A \; ;$$

$$\tan \tfrac{1}{2}(B - C) = \frac{\sin \frac{1}{2}(b - c)}{\sin \frac{1}{2}(b + c)} \cot \tfrac{1}{2} A .$$

By these we find $\frac{1}{2}(B + C)$ and $\frac{1}{2}(B - C)$.

Then $\quad \frac{1}{2}(B + C) + \frac{1}{2}(B - C) = B$

and $\quad \frac{1}{2}(B + C) - \frac{1}{2}(B - C) = C$

and the angles are found.

Of course we might find these angles by first finding the third side and thus bringing it under Case I, but this would require unnecessary work where the third side is not asked for.

Conversely, (i) of this case may be solved by finding the two remaining angles, and thus bringing it under Case II.

20. Case IV. Two angles and the included side given.

This being the polar of Case III has the same character in having two sub-cases, according as the third angle or the two remaining sides are required.

(i) *The third angle being required.*

With natural functions this is obtained by formula (48).

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

And this may be made logarithmic as follows;

$$\cos A = \cos B \sin C \{\tan B \cos a - \cot C\}.$$

Put $$\tan B \cos a = \cot \phi,$$

Then

$$\cos A = \frac{\cos B \sin (C - \phi)}{\sin \phi}$$

Or
$$\left.\begin{aligned} &\text{l. cot } \phi = \text{l. tan } B + \text{l. cos } a \\ &\text{l. cos } A = \text{l. cos } B + \text{l. sin } (C - \phi) - \text{l. sin } \phi \end{aligned}\right\} \ldots (65)$$

(ii) *The two remaining sides being required.*

This is best effected by Napier's analogies, iii and iv of (59).

$$\left.\begin{aligned} \text{l.tan}\frac{a+b}{2} &= \text{l.cos}\frac{A-B}{2} - \text{l.cos}\frac{A+B}{2} + \text{l.tan}\frac{c}{2} \\ \text{l.tan}\frac{a-b}{2} &= \text{l.sin}\frac{A-B}{2} - \text{l.sin}\frac{A+B}{2} + \text{l.tan}\frac{c}{2} \end{aligned}\right\} \quad \ldots\ldots (66)$$

EXERCISE.

1. Given $b = 128°$, $c = 150°$, $A = 36°20'$, to find the third side, and the two remaining angles.

2. Given $B = 100°$, $C = 84°$, $a = 52°40'$, to find third angle, and the two remaining sides.

3. Of three towns, D, E, F, E is 560 miles north-east from D, and F is 680 miles north-west 15° north from D. Find the distance from E to F.

21. Case V. Two sides and an angle opposite one of them.

Case VI. Two angles and a side opposite one of them.

These two cases may be considered together, as they are both solved by the sine formula.

Thus $\sin a : \sin A = \sin b : \sin B$

finds A when a, b, and B are given ; and it also

finds a when A, b, and B are given.

In both cases the side or angle to be found is determined by its sine alone, and may thus be ambiguous. This ambiguity may be removed, when possible, by the relation stated in (ii) Art. (2).

Ex. Given $a=72°45'$, $b=68°12'$, $A=124°$, to find B.

Here $\quad$ l. sin B=l. sin A +l. sin b −l. sin a,

or $\quad$ l. sin B=9·91857+9·96778+0·01999=9.90634

$$\therefore \quad B=53°42', \text{ or } 126°18',$$

and as $a>b$, so $A>B$, and $B=53°42'$.

22. Some variations.

(i) When we have two sides and the included angle given we may find another angle by the cotangent formula ; and when we have two angles and the included side given we may find another side by the same formula.

Thus if we are given a and b and the angle C we have

$$\cot a \sin b=\cos b \cos C+\sin C \cot A,$$

$$\therefore \quad \cot A=\frac{\cot a \sin b-\cos b \cos C}{\sin C}$$

which finds the angle A by natural functions.

$$\text{Similarly,} \quad \cot B=\frac{\cot b \sin a-\cos a \cos C}{\sin C}$$

The first is made logarithmic by means of an auxiliary angle as follows :

Let $\quad \cot b \cos C=\cot \theta$

Then $\quad \cot A=\dfrac{\sin b \sin (\theta-a)}{\sin a \sin \theta \sin C}$

$$\left.\begin{array}{l} \text{or} \quad \text{l. cot } \theta = \text{l. cot } b + \text{l. cos } C \\ \text{and} \\ \text{l. cot } A = \text{l. sin } (\theta - a) + \text{l. sin } b - \text{l. sin } a - \text{l. sin } \theta \\ \qquad\qquad\qquad\qquad - \text{l. sin } C \\ \text{Or} \quad \text{l. tan } A = \text{l. sin } a + \text{l. sin } \theta + \text{l. sin } C - \text{l. sin } b \\ \qquad\qquad\qquad\qquad - \text{l. sin } (\theta - a). \end{array}\right\} \ldots (67)$$

(ii) When a, b, and A are given we can find B by the sine formula ; and when A, B, and a are given we can find b by the same formula.

Then, in order to find C we resort to Napier's analogy (ii) of (59)

$$\cot \tfrac{1}{2} C = \frac{\sin \frac{1}{2}(a+b)}{\sin \frac{1}{2}(a-b)} \tan \tfrac{1}{2}(A-B).$$

And having found C we find c by the sine formula.

Ex. 1. Given $B = 52^\circ 20'$, $C = 68^\circ 15'$, $a = 72^\circ$, to find the remaining parts of the triangle.

Take form (65) Case IV.

l. tan B = 0·11241
l. cos a = 9·48998

l. cot ϕ = 9·60239
$\phi = 68^\circ 11'$
$C = 68^\circ 15'$
$\therefore\ C - \phi = 0^\circ 4'$

l. sin $(C - \phi)$ = 7·06579
l. cos B = 9·78609
col. sin ϕ = 0·03228

l. cos A = 6·88416
$\therefore\ A = 89^\circ 57'$

Then taking the sine formula—

$$\begin{aligned} \text{l. sin } b &= \text{l. sin } a + \text{l. sin } B - \text{l. sin } A \\ &= 9{\cdot}97821 + 9{\cdot}89849 + 0{\cdot}00000 \\ &= 9{\cdot}87670 \end{aligned}$$

and $b = 48^\circ 50'$. Thence in a similar way $c = 62^\circ 3'$.

Ex. 2. Given $A = 66°30'$, $B = 27°40'$, $a = 87°$, to find the remaining parts.

1. By sine formula, Case VI,

$$\begin{aligned} \text{l. sin } b &= \text{l. sin } a + \text{l. sin } B + \text{col. sin } A \\ &= \text{l. sin } 87° + \text{l. sin } 27°40' + \text{col. sin } 66°30' \\ &= 9{\cdot}70382, \end{aligned}$$

and $b = 30°22'$, as b must be less than a.

Then to find c we have from Napier's analogy (iv) Form (59)—

$$\tan \tfrac{1}{2} c = \tan \tfrac{1}{2} (a - b) \cdot \frac{\sin \tfrac{1}{2} (A + B)}{\sin \tfrac{1}{2} (A - B)}.$$

But $\frac{1}{2}(A+B) = 47°5'$; $\frac{1}{2}(A-B) = 19°25'$; $\frac{1}{2}(a-b) = 28°19'$,

$$\begin{aligned} \therefore \text{l. tan } \tfrac{1}{2} c &= 9{\cdot}73144 + 9{\cdot}86472 + 0{\cdot}47829 \\ &= 0{\cdot}07445, \end{aligned}$$

whence $\frac{1}{2} c = 49°53'$, and $c = 99°46'$.

Thence C may be found by the sine formula.

EXERCISE.

1. The latitudes of two places are given $27°30'$ N. and $38°50'$ N., and the difference of their longitudes is $10°40'$, to find the distance between the places. (Take the north pole of the earth as a vertex of the triangle).

2. P and Q are two places. P's latitude is $46°$ N., and Q's longitude is $6°20'$ greater than P's. If the distance from P to Q is 600 miles, find the latitude of Q.

3. The sides a, b and the angle B is given, to find the side c.

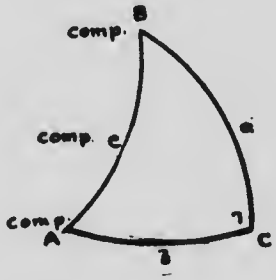
4. If ABC be a spheric triangle right-angled at C, prove the following relations—

(i) $\cos c = \cos a \cos b$ (ii) $\sin b = \sin c \sin B$
(iii) $\sin a = \sin c \sin A$ (iv) $\cot B = \sin a \cot b$
(v) $\cos c = \cot A \cot B$ (vi) $\cos A = \cos a \sin B$

23. Napier's Circular Parts.

These are a couple of rules which apply to all cases of right-angled spheric triangles.

They were discovered by Lord Napier in 1614, but how he came to their discovery is not certainly agreed upon, and no very simple proof can be given for them. The 6 relations given under Ex. 4 of the preceding Art. are directly deduceable from the forms already given by making C a right angle. But they all proceed at once from Napier's rule of circular parts.



ABC is a triangle right-angled at C. Then the right angle being ignored, we have the following, which are taken as circular parts for the application of the rules — a, b, comp. of c, comp. of A, comp. of B ; or the legs of the triangle, and the complements of the angles and hypothenuse.

Also, of any three parts under consideration, that one which holds the same relative position to each of the other two is called the **middle part**, and the other two are **adjacent parts** when they lie next the middle

part, and opposite parts when they do not. Thus c is adjacent to A, and to B, but it is opposite with respect to a since the part B lies between.

The rules are as follows :—

The *si*ne of the m*i*ddle part = the product

of { the c*o*sines of the *o*ppo*s*ite parts.
   { the t*a*ngents of the *ad**a***jcent parts.

This may be readily remembered by noticing that *i* is a prominent letter in sine and middle ; *o* is a prominent letter in cosine and opposite ; and *a* in tangent and adjacent.

Ex. 1. To find the relation between a, b, and c; c is evidently the middle part, and a and b are opposites.

$\therefore$ sin (comp. of c) = cos a · cos b; or cos c = cos a cos b. (See i of Ex. 4, Art. 22.)

Ex. 2. To find the relation between A, B, and c.

Again c is the middle part, and A and B are adjacent parts.

$\therefore$ sin (comp. c) = tan (comp. A) tan (comp. B)

or cos c = cot A cot B. (See v. Ex. 4),

EXERCISE.

1. Find the relations existing among the following—

   (i) A, B, a. (ii) A, B, b. (iii) A, a, b.
   (iv) A, a, c. (v) B, b, c.

2. In any spheric triangle, given two sides and the included angle, to find the third side by means of Napier's circular parts.

Let A, b, c be given to find a. Draw the great circle segment BD $\perp$ AC, and denote BD by p and AD by x. Then DC$=b-x$.

Now	$\sin p = \sin c \sin A$	making p known
	$\tan x = \cos A \tan c$	" x "
	$\cos a = \cos p \cos (b-x)$	" a "

This solution is thus effected by three simple equations which are logarithmic.

3. Given a, B, C, in any triangle to find A.
(Draw a $\perp$ from B on AC, or from C on AB).

4. A and C are on the same meridian 300 miles apart, and A and B are on the equator 500 miles apart. Find the distance from B to C.

24. Applications of Spheric Trigonometry.

Many of the applications of Spheric Trigonometry are to figures formed by great circles drawn on a sphere, such as the earth, or the visible surface of what is called the sky.

These great circles form spheric lines, and any three of them not passing through a common point determine a spheric triangle on the surface of the sphere. Thus any three stars, not in line, form the vertices of a spheric triangle upon the sphere of the heavens.

If such a triangle be formed upon a sphere of definite radius, the triangle has a definite area, and the expression for the area of the triangle must of necessity involve the radius of the sphere, as linear measure does not enter into the conception of the sides and angles of the triangle.

Two spheric lines intersect at opposite points on the sphere and divide the surface into four sections called *lunes* (luna, the moon). These also have a definite area if the sphere has a definite radius.

25. Theorem.

The area of a lune is equal to twice the square of the radius of the sphere multiplied by the radian measure of the angle of the lune.

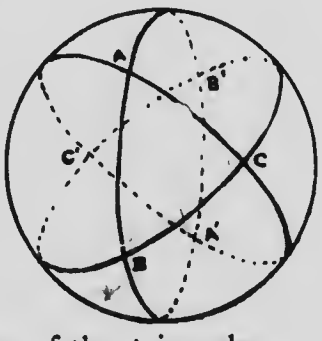
Let A be the angle measured in radians.

Then $A : 2\pi =$ area of lune : area of sphere $(4\pi r^2)$:

or area of lune $= 2Ar^2$.

26. Theorem.

The area of a spheric triangle is equal to the square of the radius of the sphere multiplied by the radian measure of the spherical excess.



The spheric lines, which form the triangle, when taken in pairs give us three lunes which each overlap the triangle and whose angles are A, B, and C. The sum of the areas of these lunes is $2r^2A + 2r^2B + 2r^2C$, and it is readily seen that the sum of the lunes is equal to one-half the surface of the sphere and twice the area of the triangle.

$$\therefore \quad 2r^2(A+B+C) = 2\pi r^2 + 2\Delta$$

$$\text{Or} \quad \Delta = r^2(A+B+C-\pi).$$

But $A+B+C-\pi$ is the spherical excess, E.

$$\therefore \quad \Delta = Er^2.$$

EXERCISE.

1. Considering the whole surface of the sphere as being included in the triangle, show that $E=4\pi$. (The points A, B, C determine the two triangles, one the smaller, and the other all that is left of the spheric surface.

   The angles of the greater triangle are $2\pi-A$, $2\pi-B$, $2\pi-C$; and the sum $-\pi$
   $=6\pi-(A+B+C)-\pi=5\pi-(A+B+C)$.

   But when the smaller triangle reduces to zero the sum of its angles is π.

   $\therefore E=4\pi$ for the whole sphere).

2. A triangle on the earth has its angles 120°, 60°, 10°, to find its area. Earth's radius=3960m.

3. The sides of a terrestrial triangle are 123, 121, and 116 miles, to find its spherical excess and thence its area.

4. If the triangle of Ex. 3 has its area calculated as a plane triangle, and its spherical excess thus calculated, find the error.

5. A triangle is equilateral and its spherical excess is known to be 12′. Find approximately its side.

27. The spherical excess plays so important a part in large geodetic surveys, that formulae have been investigated for finding it directly from the sides of the triangle. Two such formulae are prominent.

(i) Cagnoli's formula.

since $E = A + B + C - \pi$

$$\frac{E}{2} = \frac{A+B}{2} - \frac{\pi - C}{2},$$

and $$\sin\frac{E}{2} = \sin\frac{A+B}{2}\sin\frac{C}{2} - \cos\frac{A+B}{2}\cos\frac{C}{2};$$

and putting for $\sin\frac{A+B}{2}$ and $\cos\frac{A+B}{2}$ their values from (i) and (ii) of (58),

$$\sin\frac{E}{2} = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}\,\frac{\sin C}{2} - \frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}\,\frac{\sin C}{2}$$

$$= \frac{\cos\frac{1}{2}(a-b) - \cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c} \cdot \frac{\mathsf{u}}{2\sin b \sin a}$$

$$\left.\begin{aligned} &= \frac{\mathsf{u}}{4\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}}; \text{ ....} \\ &= \sqrt{\frac{\{\sin s \sin(s-a)\sin(s-b)\sin(s-c)\}}{2\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}}} \text{ .....} \end{aligned}\right\} (69)$$

The u occurring here is the same as that of (51) and (52). The formula thus established is adapted to logarithmic calculation.

(ii) **Lhuilier's formula.**

$$\frac{\sin\frac{1}{2}(A+B)-\sin\frac{1}{2}(\pi-C)}{\cos\frac{1}{2}(A+B)+\cos\frac{1}{2}(\pi-C)}=\frac{\sin\frac{1}{4}(A+B+C-\pi)}{\cos\frac{1}{4}(A+B+C-\pi)}$$

$$=\tan\tfrac{1}{4}E. \qquad \therefore \tan\tfrac{1}{4}E=\frac{\sin\frac{1}{2}(A+B)-\cos\frac{1}{2}C}{\cos\frac{1}{2}(A+B)+\sin\frac{1}{2}C}$$

And substituting for $\sin\frac{1}{2}(A+B)$ and $\cos\frac{1}{2}(A+B)$ from Delambre's theorems (58), the foregoing reduces to

$$\tan\frac{E}{4}=\sqrt{\left\{\tan\frac{s}{2}\tan\frac{s-a}{2}\tan\frac{s-b}{2}\tan\frac{s-c}{2}\right\}}. \quad (70)$$

This very neat and concise form is adapted to the use of logarithms.

Ex. Given that the sides of a spherical triangle are 38°, 46°, and 70°, to find the spherical excess.

$\frac{1}{2}s=38°30'$, $\frac{1}{2}(s-a)=19°30'$, $\frac{1}{2}(s-b)=15°30'$, $\frac{1}{2}(s-c)=3°30'$

l. tan. $\frac{1}{4}E=\frac{1}{2}(9{\cdot}90060+9{\cdot}54915+9{\cdot}44299+8{\cdot}78649)$
$=8{\cdot}83961$

$\therefore \quad \frac{1}{4}E=3°57'14''$ and $E=15°48'56''$.

THE EARTH.

28. The earth is approximately a sphere, and for general purposes may be treated as a sphere ; and for this reason we shall first treat of a spherical earth.

The earth rotates about a diameter called the axis, and the end points of the axis are the poles, one being the north pole, and the other the south.

Any great circle passing through the poles is a meridian, and every place upon the earth's surface, except the poles, has its own meridian.

The great circle equidistant from the poles is the equator. All smaller circles drawn to the same poles are circles of latitude. The equator is the circle of zero-latitude, and each circle of latitude cuts all the meridians orthogonally.

The latitude of a place is the angular distance of the place from the equator, measured along the meridian of the place. Latitude is thus either north or south, but it is convenient, in many cases, to consider south latitude as being negative. With this view the complement of the latitude is the north polar distance.

The longitude of a place is the angular distance, measured along the equator, between the meridian of the place and some meridian taken as a first or prime meridian. In English speaking countries the meridian of the Royal Observatory, at Greenwich, near London, is usually taken as the first meridian.

Size of the Earth.

29. In some level country a base line must be measured along a meridian from north to south, or south to north, as accurately as possible, as any error in this

measurement will be magnified manifold in the earth's radius. Such base lines have been measured, with all possible care, in many different places on the earth's surface.

Suppose this base line to be 16.7158 miles.

At the extremities of this line observations are made upon the altitude of *Polaris* or the north star.

Suppose again that at the south end the altitude of Polaris proves to be $41°46'38''·8$, and at the north end of the line $42°1'10''·3$.

Then the angle formed by lines drawn from these extremities to the centre of the earth, or the angle subtended at the centre by the base line is

$$14'31''·5 = 0°·2421 = 0·00422 \text{ radians.}$$

$$\therefore \quad r = \frac{s}{\theta} = \frac{16·7158}{·00422} = 3961·1 \text{ miles.}$$

This is only an approximation, as all the radii are not quite the same. But a mean value may be taken as **3960** miles.

Multiplying this by $2\pi = 6·2831852$, we get the circumference of the earth as 24881·4 miles.

Then, dividing by 360, and again by 60, we get the following—

$1° = 69·11$ miles
$1' = 1·1518$ " $= 6081·6$ feet.
$1'' = 0·01919$ " $= 101·33$ "

and thence

$$1 \text{ mile} = 0°·01447 = 0'·868 = 52''·1.$$

Hence if m denotes a distance in miles upon the earth's surface measured along a great circle,

0·01447 m is the distance in degrees ; and

0·8682 m is the distance in minutes.

Also, if μ° denotes a distance in degrees, 69·11 μ is the distance in miles ; and if μ' denote a distance in minutes, 1·1518 μ is the distance in miles.

Ex. 1. Two places on the same meridian are 325 miles apart, to find the difference in their latitudes.

$$325 \times 0{\cdot}01447 = 4^\circ 703 = 4^\circ 42' \text{ nearly.}$$

In nautical practice, a practice that can be profitably introduced in some other places, the minute of arc is taken as a unit-length and is called a *knot*. Thus 60 knots is equal to 69.11 miles, and 1 knot = 1.1518 miles, and 1 mile = 0.8682 knots.

EXERCISE.

1. Travelling from the equator northwards 1672 miles brings you into what latitude?

2. What is the angle between two plumb-lines 3 miles apart?

3. A canal is straight and 14 miles long; what is the angle between the levels at the ends of the canal?

4. If a ship sails 480 knots, how many miles does it go?

5. The lat. of Washington, D.C. is 38°53′ N., and of Kingston it is 44°13′ N., and they are on the same meridian. Find their distance apart in miles.

Direction.

30. Direction on the earth's surface is a somewhat indefinite idea. At the north pole there is only one direction, south; and at the south pole all directions are north. At the equator directions are fairly consistent.

Properly speaking, direction at a point should be along a great circle, it being the analogue of the straight line in the plane, and the meridian, at the point under consideration, should be the line of reference.

Thus the direction of B from A, or the direction *from A to B*, is the dihedral angle between the plane of the meridian of A and the plane of the great circle through A and B. It thus appears that if A and B are not both on the equator, or both on the same meridian, the direction of B from A is not the opposite of the direction of A from B. And a course directed along a great circle which is neither the equator nor a meridian is continually changing its direction.

Ex. Given two places, A, lat. N. 36°20′, long. W. 68°; and B, lat. N. 62°30′, long. W. 125°, to find the direction from A to B, and also the direction from B to A.

With P as north pole we have the triangle APB, in which PA = 53°40′, PB = 27°30′, $\angle P = 57°$, or two sides and the included angle. Hence by Napier's analogies

$$\tan \tfrac{1}{2}(B+A) = \frac{\cos 13°5'}{\cos 40°35'} \cot 28°30' = 0{\cdot}64086$$

$$\tan \tfrac{1}{2}(B-A) = \frac{\sin 13°5'}{\sin 40°35'} \cot 28°30' = 2{\cdot}36217$$

$$\therefore \quad \tfrac{1}{2}(B+A)=67^\circ 3', \ \tfrac{1}{2}(B-A)=32^\circ 39'.$$

$$\therefore \quad B=99^\circ 42', \ A=34^\circ 24'.$$

Hence the direction from A to B is north $34^\circ 24'$ west, and from B to A it is south $80^\circ 18'$ East, or East $9^\circ 42'$ south.

EXERCISE.

1. Starting from the equator in a north-east direction, we go 800 miles on a great circle. In what direction are we going at the terminal?
2. At latitude 38° N. we run a great circle line east for 50 miles. What is our direction at the termination of the line?
3. Starting from latitude 42° N. and long. 75° W. we follow a great circle in a direction east 36° north. Find the latitude and longitude of the point at which we will cross a meridian at right angles.

DISTANCE.

31. Properly speaking the distance from point to point on the surface of a sphere should be measured along the shortest path from the one point to the other, that is along the great circle joining the points. But it is convenient at times to speak of the distance as being measured along some other curve, such as a parallel of latitude, or along a curve called the *loxodrome* or *rhumb line.*

The great circle. If the angle, subtended at the earth's centre, by the arc of the great circle joining two points, be denoted by θ°, the distance between the points, measured along the great circle, is $69.1\ \theta$ miles, where θ is taken in degree measure.

If θ be in radians, dist. $= 69{\cdot}1 \frac{180}{\pi} \theta$ miles $= 3960\,\theta$ miles.

$$\left.\begin{array}{l} \text{Hence } \theta \text{ in degrees—distance} = 69{\cdot}1\,\theta \\ \theta \text{ in radians—} \quad \text{“} \quad = 3960\,\theta \end{array}\right\} \ldots\ldots(71)$$

Ex. To find the distance from Kingston to St. Louis, the latitudes being 44°13′ N. and 38°36′ N., and the difference in longitude being 13°50′.

This problem requires us to find the third side of a triangle when two sides and the included angle are given.

The sides are 45°47′ and 51°24′, and the angle 13°50′. This falling under Case III we have

$$\cos c = \cos 45°47' \cos 51°24' + \sin 45°47' \sin 51°24'$$
$$\cos 13°50' = {\cdot}62388 \times {\cdot}69737 + {\cdot}71671 \times {\cdot}78152 \times {\cdot}97100$$
$$= {\cdot}97897 = \cos 11°46'$$

And $11°46' \times 69{\cdot}1 = 813$ miles nearly.

EXERCISE.

1. Find the distance from Chicago to Cincinnati, Chicago being 41°50′ N. 5h. 50m. 27s. W., and Cincinnati 39°6′ N. and 5h. 37m. 59s. W.
2. *A's* lat. is 44°13′ N. and long. 5h. 5m. 55s. W., and *B* is 360 miles from *A* at lat. 46°10′ N. Find *B's* longitude.
3. Find the distance in miles from lat. 0, long. 0, to lat. 45, long. 180.
4. Starting from lat. 0, long. 0, at angle W. 30° N., and going 6000 miles, What would then be our latitude and our longitude?

32. Distance along a parallel of latitude.

Sometimes it is convenient to express distance as measured along a parallel of latitude instead of along a great circle. Of course this can apply only to places having the same latitude.

The radius of a circle of latitude considered as a plane circle is $r \cos \phi$, where r is the earth's radius and ϕ is the latitude.

And as the lengths of similar arcs are as the radii of the circles of which they form a part, it follows that the distance between two meridians, measured along a parallel of latitude is found by multiplying the corresponding distance on the equator by $\cos \phi$. Or if two places on the same parallel differ by $p°$, their distance apart in miles measured along their common parallel is

$$69{\cdot}1\, p \cos \phi \ldots \ldots \ldots \quad \ldots \ldots (72)$$

Ex. A person follows the parallel of 44°13′ N. from long. 76°20′ to long. 84°12′, to find the length of the journey.

Here $p = 7°52' = 7°{\cdot}867$ and $\cos 44°13' = {\cdot}71671$

$\therefore$ Distance $= 69{\cdot}1 \times 7{\cdot}867 \times {\cdot}71671 = 389{\cdot}61$ miles nearly.

Ex. In travelling 600 miles along parallel 50°40′ N., by how much do you change your longitude?

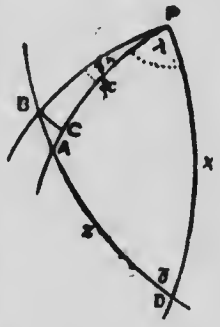
33. The loxodrome or rhumb line.

The mariner's compass is divided into four principal parts, north, east, south, and west. Each of these is then divided into eight parts called a *rhumb*, and each rhumb is further divided into four parts called quarter rhumbs. Thus each quarter rhumb is equal to 2°48′45″.

A line on the earth's surface which cuts every meridian at the same rhumb, that is, the same angle, the right angle being excluded, is a rhumb line. It is called the loxodrome from λοξος, slanting.

This curve is somewhat common in navigation, and is followed when a ship is steered to a fixed point of the compass, and it offers so many advantages that navigator's charts are usually constructed with reference to it.

The loxodrome is a curve or spiral upon the surface of a sphere. But as it is not a great circle, or any part of one, it cannot be dealt with by the formulae of spheric trigonometry. Its full consideration requires the principles of the calculus.



Let P be the pole, D be the place from which the ship starts, called the point of departure, and A be the point of arrival. PD and PA are the meridians of the points of departure and arrival respectively, and $\lambda = \angle APD$ is the difference in longitude. Let X and x denote respectively the north-polar-distances of the points of departure and arrival, and let z denote the distance from D to A, or the total run. Produce the loxodrome, DA, to B, draw the meridian PB, and the parallel of latitude BC. Then $CPB = \delta\lambda$ is the increment of λ; $AC = \delta x$, being the difference between PA and PB, is the increment of x; and $AB = \delta z$ is the increment of z.

Then in the triangle BCA, $\angle BCA$ is a right angle, $\angle CAB = \angle PDB = \gamma$, as the loxodrome cuts all the meridians at the same angle. And we have approximately, $AB = AC \sec \angle BAC$, or $\delta z = \delta x \sec \gamma$.

And going to the limit,

$$dz = dx \sec \gamma.$$

Whence, integrating from X to x we get

$$\left.\begin{aligned} z &= (X - x) \sec \gamma \\ \text{Or} \qquad z &= (l - l_0) \sec \gamma \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (73)$$

where l_0 is the latitude of departure and l is the latitude of arrival.

If x and X, or l and l_0 be in degrees, z will also be in degrees.

Ex. A ship sails from lat. 23°27′ N. to 44°13′ N. on a course N.E. 5° north. Find the distance sailed.

Here $l_0 = 23°27'$ N., $l = 44°13'$ N., and $l - l_0 = 20°46'$ and $\gamma = 40°$. $\therefore z = 20°{\cdot}7667 \times \sec 40° \times 69{\cdot}1 = 1873$ miles.

34. From the foregoing figure we have also, approximately,

$$BC = AC \tan BAC, \text{ or } \delta\lambda \sin x = \delta x \tan \gamma.$$

or going to the limit

$$d\lambda = \frac{dx}{\sin x} \cdot \tan \gamma.$$

Hence, integrating from X to x we get

$$\lambda = \left\{\log. \frac{\tan \frac{1}{2} x}{\tan \frac{1}{2} X}\right\} \tan \gamma.$$

In this formula the angle λ is in radians and the logarithm is Naperian. We change to degrees and decimal logarithms as follows :

Denote the degrees in λ by L, and decimal logarithms by Log. Then

$$\text{L}\cdot\frac{\pi}{180}=\left\{\text{Log}\frac{\tan\frac{1}{2}x)}{\tan\frac{1}{2}X)}\right.\frac{\tan\gamma}{0\cdot43429}$$

$$\text{or}\quad \text{L}=131\cdot93\ldots\times\left\{\text{Log}\frac{\tan\frac{1}{2}x}{\tan\frac{1}{2}X}\right\}\tan\gamma\ldots\ldots\ldots(74)$$

This formula finds the change in longitude in terms of the latitudes of the departure and arrival, and the meridional angle of the course.

Ex. A ship sails on a rhumb line 45° from N. from N.P.D. 44°13′ to N.P.D 30°, to find the change in longitude..

$\frac{1}{2}x=15°$, and $\frac{1}{2}X=22°6'\frac{1}{2}$, $\tan\frac{1}{2}x=0\cdot26795$, $\tan\frac{1}{2}X=0\cdot40623$, $\tan\gamma=1\cdot0000$

$$\text{Log}\frac{\tan\frac{1}{2}x}{\tan\frac{1}{2}X}=\text{Log}\frac{26795}{40622}=-0\cdot18072$$

$$\therefore\quad \text{L}=-0\cdot18072\times(1)\times131\cdot93=-23°\cdot843$$
$$=-23°50'\text{ nearly.}$$

Otherwise, l. tan 15° − l. tan 22°6½′ = −0·18072, and then as before.

EXERCISE.

1. A ship starting from lat. 16° N. and long. 170° W. sails on the rhumb line N.E. 10°E. until it reaches 162°W. What is her latitude then?
2. Find the course along a loxodrome from Sandy Hook, 40°25′ N., 73°52′ W. to Southampton, 50°54′N., 1°23′W.
3. If l_0 be the latitude of departure and l be that of arrival,

$\text{L}=131\cdot93\tan\gamma\log\{(\sec l-\tan l)(\sec l_0-\tan l_0)\}$.

4. If a ship could start from the equator and following a rhumb line of 45° reach the pole, prove that the distance travelled would be 8795 miles, and that the change in longitude would be infinite.

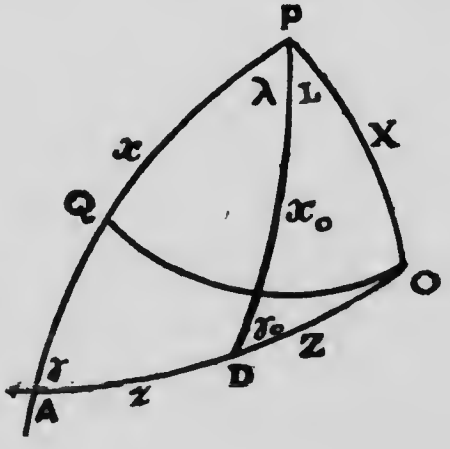
5. Find the difference in distance from lat 22° N. to lat. 45°N., and respective longitudes 0° and 25° W. when following the great circle and when following the loxodrome.

6. The Canada-American boundary line follows the 49th parallel of latitude from long. 94°20′ W. to 124°30′ W. Find the length of this boundary.

35. To survey a great circle.

The great circle is the course usually called a "line" on the earth's surface. A line run by means of vertical stakes is nominally a great circle. But if the line is to be of any considerable length, unavoidable errors will creep into the survey, and unless corrections are applied at proper intervals the line may come to differ considerably from a great circle.

Let DA be part of a great circle, D being the point of departure and A the point of arrival. At some point, O, this great circle cuts a meridian at right angles. Let P be the north pole of the earth, and PO, PD, PA be meridians, and let OQ be the parallel of latitude through O.



Denote PO by X, PD by x_0, PA by x, OD by Z, DA by z, $\angle$ PDO by γ_0, and $\angle$ PAO by γ.

Then POD and POA are triangles right-angled at O. Hence applying Napier's circular parts—

1. Sin X = sin x_0 sin γ_0.

2. Sin Z = tan X cot γ_0.

These serve to find X and then Z in terms of the known quantities x_0 and γ_0, which denote the north polar distance of the point of departure, and the angle which the course makes with the meridian.

The distance z is known by being measured in the survey. Then $Z+z$ is known, and

3. cos x = cos X cos $(Z+z)$.

4. tan γ = tan X cosec $(Z+z)$.

3 and 4 make known x and γ, the north-polar distance of the point of arrival, and the angle which the course makes with the meridian at this point.

And these as obtained by observation should agree with those resulting from the calculation of the survey.

Again, denoting $\angle$ OPD by L, and $\angle$ DPA by λ,

5. cos L = tan X cot x_0.

6. cos $(L+\lambda)$ = tan X cot x.

These give the means of finding λ, the change in longitude in passing from D to A.

Collecting we have in logarithmic form

$$\left.\begin{array}{l} 1.\ \text{l. sin } X = \text{l. sin } x_0 + \text{l. sin } \gamma_0 \\ 2.\ \text{l. sin } Z = \text{l. tan } X + \text{l. cot } \gamma_0 \\ 3.\ \text{l. cos } x = \text{l. cos } X + \text{l. cos } (Z+z) \\ 4.\ \text{l. tan } \gamma = \text{l. tan } X + \text{col. sin } (Z+z) \\ 5.\ \text{l. cos } L = \text{l. tan } X + \text{l. cot. } x_0 \\ 6.\ \text{l. cos } (L+\lambda) = \text{l. tan } X + \text{l. cot } x \end{array}\right\} \quad \ldots\ldots\ldots (75)$$

Ex. A course starts at 36° S. of W. in lat. 44°13′ N., long. 76°20′ W., and follows a great circle for 300 miles. To find the latitude, the longitude and the angle with the meridian at the point of arrival.

Here $x_0 = 45°47'$, $\gamma_0 = 54°$, $z = 4°20'{\cdot}5$.

1. l. sin $x_0 = 9{\cdot}85534$
l. sin $\gamma_0 = 9{\cdot}90796$

l. sin $X = 9{\cdot}76330$

2. l. tan $X = 9{\cdot}85228$
l. cot $\gamma_0 = 9{\cdot}86126$

l. sin $Z = 9{\cdot}71354$
$Z = 31°8'$
$z = 4°20'{\cdot}5$

$Z + z = 35°28'{\cdot}5$

3. l. cos $X = 9{\cdot}91102$
l. cos $(Z+z) = 9{\cdot}91082$

l. cos $x = 9{\cdot}82184$
$x = 48°26'$

4. l. tan $X = 9{\cdot}85228$
col. sin $(Z+z) = 0{\cdot}23632$

l. tan $\gamma = 0{\cdot}08860$
$\gamma = 50°48'$

5. l. tan $X = 9{\cdot}85228$
l. cot $x_0 = 9{\cdot}98812$

l. cos $L = 9{\cdot}84040$

$\therefore\ L = 46°10'$

6. l. tan $X = 9{\cdot}85228$
l. cot $x = 9{\cdot}94783$

l. cos $(L+\lambda) = 9{\cdot}80011$
$L + \lambda = 50°52'$

$\therefore\ \lambda = 4°42'$.

Hence, the new latitude is $41^\circ 34'$ N., the new longitude is $76^\circ 20' + 4^\circ 42' = 81^\circ 2'$, and the new angle with the meridian is $50^\circ 48'$.

If the survey is to be checked by finding the latitude of the stations at every 10 miles, say, we may do as follows: For a fixed point of departure and a fixed direction at that point, X and Z are constant, while z varies as we proceed. Now 10 miles $= 8'41''$,

$\therefore$	10 m.	20 m.	30 m.	40 m.	50 m.
l. cos	$Z + 8'41''$	$Z + 17'22''$	$Z + 26'3''$	$Z + 34'44''$	$Z + 43'25''$
l. cos X	....	....	....	....	....
l. cos x	....	....	....	....	....

and this gives us the new latitude at every 10 miles.

Similarly

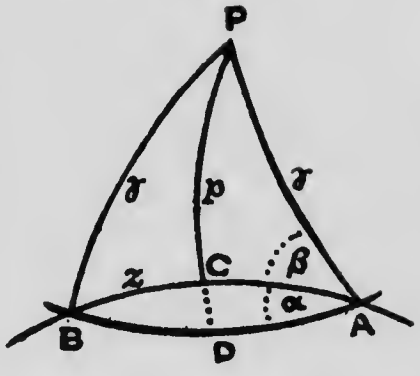
col. sin	$Z + 8'41''$	$Z + 17'22''$	$Z + 26'3''$	$Z + 34'44''$	$Z + 43'25''$
l. tan X	....	....	....	....	....
l. tan γ	....	....	....	....	....

and this gives the new direction at every 10 miles.

Any difference between the quantities so found and the results of the practical survey indicates some error in the carrying out of the practical operations.

36. To survey a parallel of latitude.

A line run by a series of stakes, each perpendicular to the earth's axis, would be a parallel of latitude. But as such a method is not practicable, we must try and survey a spheric polygon whose sides are not more than 5 or 10 miles each, and whose vertices lie on the required parallel.



A and D are points on the parallel of latitude, ADB, and ACB is the great circle through A and B, and PC is ⊥ to AB.

Then PA = PB = γ = the colatitude of the required parallel; $\angle$ PAC + $\angle$ CAD, or $\beta + \alpha = \frac{1}{2}\pi$.

Then $\sin \alpha = \cos \beta = \cot \gamma \tan \frac{1}{2} z = \tan l \tan \frac{1}{2} z$, where l is the latitude and z is the side, ACB, of the iosceles triangle ACBP.

But, as z is a very small angle in all practical cases, we may take

$$\tan \tfrac{1}{2} z = \tfrac{1}{2} z + \tfrac{1}{3} (\tfrac{1}{2} z)^3 + \ldots$$

$$\therefore \quad \sin \alpha = \frac{z}{2} \left\{ 1 + \frac{z^2}{12} \right\} \tan l \ldots\ldots\ldots\ldots\ldots (76)$$

z must be in radians, and usually the first term of the bracket is sufficient.

Taking the first term as sufficient and denoting the number of miles in z by m, as α is a small angle we may write

$$a = \left\{ \frac{\frac{1}{2}m}{3960} \right\} \tan l$$

where a is in radians. Reducing a to minutes gives us after some reduction

$$a' = 0{\cdot}434\ m \tan l \ldots\ldots\ldots\ldots (76')$$

Ex. 1. If $l = 49°$, the parallel forming the international boundary between Canada and the United States, and m be 10 miles, this latter formula gives $0{\cdot}434 \times 10 \times 1{\cdot}15037 = 4'{\cdot}99$, or $a = 5'$ very nearly. So that starting from A and going 10 miles to B, a deflection of 10′ must be made towards the pole, and this must be repeated at the close of every run of 10 miles.

Ex. 2. Let it be required to find the distance DC, or the greatest deflection of the great circle from the circle of latitude.

In the triangle PCA, $\cos p = \cos \gamma \sec \frac{z}{2}$

But p is less than γ by a very small angle which we may denote by x and we may write

$$\cos(\gamma - x) = \cos\gamma + \sin x \sin\gamma = \cos\gamma \sec \tfrac{1}{2} z,$$

$$\text{whence } \sin x = \cot\gamma\ (\sec \tfrac{1}{2} z - 1).$$

But, as $\frac{1}{2} z$ is a very small angle, we may put

$$\sec \tfrac{1}{2} z = \sqrt{1 + \tan^2 \frac{z}{2}} = 1 + \frac{z^2}{8},$$

and $$\therefore \quad x = \frac{z^2}{8} \cot\gamma\ ; \ldots\ldots\ldots\ldots\ldots\ldots (77)$$

where x and z are both in radians.

Whence reducing x to seconds gives

$$x'' = \cdot 0016\ m^2 \tan l \quad \ldots\ldots\ldots\ldots\ldots\ldots (77')$$

And putting $m = 10$ miles and $l = 49°$ gives

$$x'' = 0''\cdot 185 = \text{about } 18\cdot 7 \text{ feet.}$$

EXERCISE.

1. The point of departure is at lat. 32° N., and the course starts at N. 16° E. and follows a great circle. Find the latitude and the direction of the course at every 15 miles, for 5 stations.

2. A great circle line starts N.W. at 64° from the pole and continues until it is 60° from the pole. Find the course at the point of arrival, and also the length of the line.

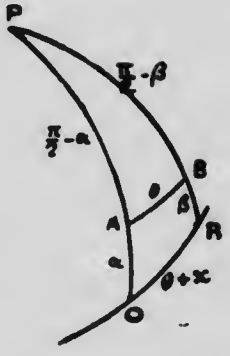
3. A certain township has two sides upon two meridians one degree apart, and two sides on the parallels of latitude 49° and 49°25′, both north. Find (1) the difference in the lengths of the north and south boundaries ; (2) the number of square miles in the township.

4. Find the angle of deflection at each 20 knots in surveying the 62nd parallel of latitude. Also find how far the survey departs from the parallel at most.

5. A great circle line starts at angle N.W., from the equator, in long. 0°, and continues for 3000 miles. Find the latitude, the course, and the longitude at the point of arrival.

6. The tropic of Cancer is the parallel 23°27′ N. Find the deflection required every 5 miles in order to follow it.

37. Reduction of an angle to the horizon.



Let A and B be two points at distances α, β, from the horizon. The angle $AB=\theta$ is measured, and it is required to find its value QR when reduced to the horizon.

In the triangle PAB where P is the zenith,

$$\cos P=\frac{\cos\theta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta} \quad \ldots\ldots\ldots \quad \ldots(78)$$

and P, being the angle measured by QR, is the value of θ when reduced to the horizon.

In practical cases α and β are small, and P exceeds θ by a very small angle, x, so that as an approximation we may put $\sin x=x$, $\cos x=1$, $\sin\alpha=\alpha$, $\sin\beta=\beta$, $\cos\alpha=1-\frac{1}{2}\alpha^2$, $\cos\beta=1-\frac{1}{2}\beta^2$.

Then

$$\cos(\theta+x)=\cos\theta-x\sin\theta=\frac{\cos\theta-\alpha\beta}{1-\frac{1}{2}(\alpha^2+\beta^2)},$$ rejecting all powers above the 2nd.

Thence $$x=\frac{2\alpha\beta-(\alpha^2+\beta^2)\cos\theta}{2\sin\theta\left\{1-\frac{1}{2}(\alpha^2+\beta^2\right\}}.$$

And $\frac{1}{2}(\alpha^2+\beta^2)$, being very small may be rejected from the denominator as compared with unity, giving

$$x=\frac{2\alpha\beta-(\alpha^2+\beta^2)\cos\theta}{2\sin\theta},$$

And finally, writing $\alpha\beta\left\{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\right\}$ for $\alpha\beta$, and $\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}$ for $\cos\theta$, this reduces to

$$x=\tfrac{1}{4}\left\{(\alpha+\beta)^2\tan\frac{\theta}{2}-(\alpha-\beta)^2\cot\frac{\theta}{2}\right\} \quad \ldots\ldots\ldots \quad (78')$$

In this formula x, α, β are all in radians. If they are to be in degrees, we obtain by a proper reduction—

$$x^\circ=0{\cdot}00438\left\{(\alpha+\beta)^2\tan\frac{\theta}{2}-(\alpha-\beta)^2\cot\frac{\theta}{2}\right\}.$$

A SPHEROIDAL EARTH.

38. If the earth were a perfect sphere all its radii would be equal, and every arc of 1° would be equal. But this is not the case, as appears from the following carefully measured arcs:

Country.	Lat.	Feet in 1°	Country.	Lat.	Feet in 1°
Sweden	66°20′	365782	America	39°12′	363786
England	52°35′	364971	India	16°8′	363004
France	44°51′	364535	Peru	1°39′	362808

These show that the curvature of a meridional section grows less as we approach the pole, and that the section is elliptic. From these by interpolation we may obtain the probable number of feet in 1° at the equator to be 362740, and at the pole 366410.

Now if N be the number of feet in 1°, and R be the corresponding radius of curvature in miles,

$$R=\frac{180\,N}{5280\,\pi}=\frac{3}{88}\cdot\frac{N}{\pi}, \text{ or}$$

$$\log.\ R=\log.\ N+8{\cdot}035488-10.$$

Hence if N_a be the number of feet in 1° at the equator, and N_b at the pole, and if R_a, R_b be the corresponding radii of curvature,

$$\log. R_a = \log. N_a + 8{\cdot}035488 - 10$$
$$\log. R_b = \log. N_b + 8{\cdot}035488 - 10$$

But from a property of the ellipse—if a and b be the equatorial and polar radii respectively,

$$a = (R_a R_b^2)^{\frac{1}{3}},\ b = (R_b R_a^2)^{\frac{1}{3}};$$

whence by substitution

$$\log. a = \tfrac{1}{3}(\log. N_a + 2 \log. N_b) + 8{\cdot}035488 - 10$$
$$\log. b = \tfrac{1}{3}(\log. N_b + 2 \log. N_a) + 8{\cdot}035488 - 10$$

Substituting for N_a and N_b their respective values, 362740 and 366410, and performing the calculation, we get

$$\left.\begin{aligned} a &= 3962{\cdot}8 \text{ miles} \\ b &= 3949{\cdot}5 \text{ miles} \end{aligned}\right\} \ldots\ldots\ldots\ldots (79)$$

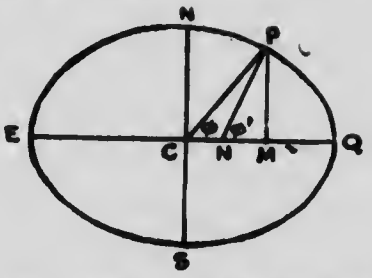
giving a difference of 13.3 miles. So that the earth's polar diameter is less than the equatorial by 26·6 miles.

39. Eccentricity.

The meridional section of the earth being an ellipse the angle between the minor axis and the focal line to the end-point of the minor axis may be called the angle of eccentricity. Denoting this angle by E, and the eccentricity by e, we have

$$\left.\begin{aligned} \cos E = \frac{b}{a} &= 0{\cdot}99664 \\ E &= 4^\circ 41' 44'' \\ \sin E = e &= 0{\cdot}08186 \end{aligned}\right\} \ldots\ldots\ldots\ldots (80)$$

40. Geocentric and apparent latitude.



The apparent latitude of a place on the earth is the angle between the direction of the plumb-line at the place, and the plane of the earth's equator.

The geocentric latitude is the angle between the earth's radius drawn to the place and the plane of the earth's equator. And the earth's meridian being an ellipse these two angles are not the same.

Let P be a given place and NPQS be the meridian passing through it. Then PC being the earth's radius to P and PN being the direction of the plumb-line and ECQ a section of the plane of the equator, $\angle PNQ=\varphi'$ is the apparent latitude, and $\angle PCQ=\varphi$ is the geocentric latitude.

Then $$\frac{\tan\varphi}{\tan\varphi'}=\frac{PM}{CM}\cdot\frac{NM}{PM}=\frac{CM-CN}{CM}=1-\frac{CN}{CM}.$$

But by a property of the ellipse $CN=e^2\cdot CM$,

$$\left.\begin{aligned}&\therefore\quad \tan\varphi=(1-e^2)\tan\varphi'=\cos^2 E\tan\varphi'\\ &\text{and } \tan\varphi'=\sec^2 E\tan\varphi.\end{aligned}\right\}\ldots.(81)$$

41. The difference between these latitudes, or the correction by which you change the one into the other is called the *angle of the vertical*, and is represented by $\angle CPN=\zeta$.

$$\tan\zeta=\tan(\varphi'-\varphi)=\frac{\tan\varphi'-\tan\varphi}{1+\tan\varphi'\tan\varphi}$$

$$=\frac{\sin^2 E\tan\varphi'}{1+\cos^2 E\tan^2\varphi'}$$

And since cos E is nearly 1 while sin E is very small, this is approximately,

$$\tan \zeta = \sin^2 E \sin \varphi' \cos \varphi' = \tfrac{1}{2} e^2 \sin 2\varphi' \ldots\ldots\ldots (82)$$

Ex. 1. Show that the maximum angle of the vertical is about 11′31″.

Ex. 2. The observed latitude of a place being 30°20′ N., find its geocentric latitude.

42. The earth's radius vector.

The equation of the ellipse is

$$b^2x^2 + a^2y^2 = a^2b^2,$$

and writing in this $x = r\cos\varphi$, $y = r\sin\varphi$ we obtain

$$r = \frac{a \sec \varphi}{\sqrt{1 + \sec^2 E \tan^2 \varphi}} \ldots\ldots\ldots\ldots (83)$$

which gives the radius vector in terms of the geocentric latitude.

To calculate r we may do as follows:

Put 1. tan θ = 1. tan φ + col. cos E,

Then 1. r = 1. a + 1. sec φ + col. sec. θ.

Ex. Calculate the earth's radius vector at lat. 60°.

Ex. Calculate the radius vector at apparent latitude 44°13′.

Exercise.

1. Two marks are 1 mile distant. One is on the horizon, and the other is 100 feet above the horizon. The angle between them is 8°10′. Find their difference in azimuth.

2. Determine the angle of the vertical at lat. 38°.

3. Find the number of miles in 1° of longitude at latitude 50°.

4. Express the angle of the vertical in terms of the geocentric latitude (φ).

5. Find an expression for the earth's radius vector in terms of the apparent latitude (φ').

6. Find the difference between the total length of the 49th parallel of latitude as it is, and as it would be if the earth were a sphere of radius a.

7. Find the difference between an arc of 50 miles on the earth and the chord of the 50 miles, the earth's radius being 3960 miles.

APPLICATIONS TO ASTRONOMY.

43. There are three important great circles in the heavens, the celestial horizon which is the horizon of the observer extended outwards to meet the celestial sphere, the celestial equator which is the terrestrial equator extended in a similar manner, and the ecliptic which is the apparent path of the sun's centre during the year, or, in other words, the earth's orbit about the sun as represented on the celestial sphere.

Each of these great circles has its poles (Art. 4) and forms the equator to its poles, and a pole and its equator gives us a system of spherical or angular coordinates.

Thus, I. *The horizontal system.*

The horizon SENW is the equator of the system, and the zenith, Z, and the nadir, Na are its poles. Any great circle passing through Z and Na is a *vertical*

circle, because it rises vertically from the horizon. But the vertical circle which passes through the north and south points of the horizon, N and S, is the *celestial meridian* of the place of observation, and its projection on the earth is the *terrestrial meridian*. Also the vertical circle which passes through E and W, the east and west points of the horizon, is the *prime vertical*.

If B is a heavenly body, as the sun or a star, and ZBH is the vertical circle through B, then HB, measured in angle is the *altitude* of B, generally denoted by a; and BZ, the complement of a, is the *zenith distance* of B, and is denoted by z.

Thus $$a+z=\tfrac{1}{2}\pi=90^\circ.$$

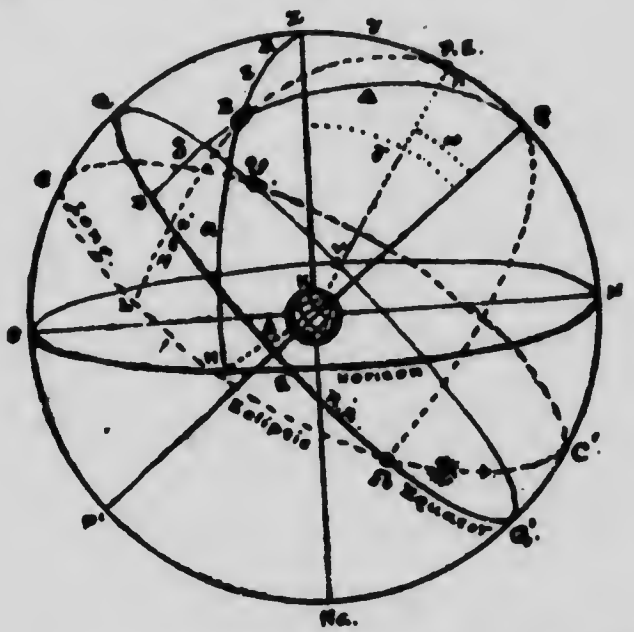


Fig.

Also, the dihedral angle between the plane of the vertical circle through B and the plane of the meridian is

the *azimuth* of B. This is the angle A in the figure if measured from the south meridian, and the supplement of A if measured from the north.

The altitude and the azimuth of a heavenly body form its coordinates in the horizon system.

The small circles parallel to the horizon are *circles of altitude*, or *parallels of altitude* or *almucanters*, as all points lying on any one of these circles have the same altitude.

The surveyor's theodolite or transit is fundamentally an *altazimuth* instrument, that is, an instrument for measuring altitudes and azimuths.

Rotation about the horizontal axis carries the point of sight along a vertical circle, and readings in this rotation give altitude, while rotation about the vertical axis carries the point of sight along a parallel of altitude, and readings give azimuth.

II. *The Equator System.* In this system the celestial equator is the equator, and the celestial poles, known as the north pole and the south pole, are the poles. Thus in the figure P and P′ are the poles, and QEQ′W is the equator. Any great circle passing through P and P′ is a meridian, and the great circle which passes through Z also is the celestial meridian of the place. If B be a heavenly body and PBD be the meridian passing through it, DB, measured in angle, is the declination, δ, of the body, and BP, the complement of δ, is the north-polar-distance, Δ.

$$\therefore \qquad \Delta + \delta = \tfrac{1}{2}\pi = 90^\circ.$$

Declination may be north or south, according as a body is on the north or the south side of the equator,

but the better usage is to give south declination a negative sign, and then polar distance will always be measured from the north pole. Thus if the sun is 12° south, or $-12°$ declination, its north-polar-distance, N.P.D., is $90° + 12° = 102°$.

The dihedral angle between the plane of a particular meridian, taken as a first meridian, and the plane of the meridian through B, is the *right ascension* of B, denoted by R.A. or by α. Right ascension is measured from the first meridian in one direction throughout the circumangle; from 0° to 360°, or from $0^h.\ 0^m.\ 0^s.$ to $24^h.\ 0^m.\ 0^s.$ if measured in time as it usually is. Of any two stars the one whose meridian lies east of that of the other has the greater right ascension.

Declination and right ascension in the heavens exactly correspond to latitude and longitude on the earth; and just as a place is registered in a geographical gazeteer by giving its longitude and its latitude, so objects in the heavens, the stars, for instance, are registered in astronomical catalogues by giving their right ascensions and their declinations.

The first meridian passes through that point where the sun crosses the equator coming northward, on or about March 21st. This is the ascending node, and the point where the sun crosses the equator going southward, on or about Sept. 23rd, is the descending node.

The ecliptic cuts the equator at these nodes. It is represented in the figure by $C\Omega C'\gamma$, and the nodes are at Ω and γ.

III. *Ecliptic System.* The pole of the ecliptic is P.E., and P.E.BL is a meridian of the ecliptic, or a

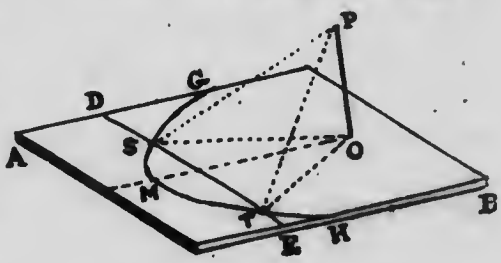
circle of longitude, the angle Ω L being the longitude of B, and LB its latitude.

The triangle BZP is of particular importance. Its sides are z, γ, Δ, or the complements of a, l, δ. The angle at P is the hour-angle, h.

PROBLEMS IN ASTRONOMY.

44. *To find the meridian.* The meridian is found when we obtain a north and south line upon the earth's surface, or when we so adjust an altazimuth instrument that its line of sight moves in the plane of the celestial meridian.

I. A crude method. A smooth rectangular board, AB, is to be placed with its edge EB somewhere nearly north and south. A compass test is sufficient. The board is to be carefully leveled and a pin or style, OP, is set up vertically near the south end of the board.



GH is a part of a circle drawn about O as centre. The pin, OP, casts a shadow, which, as the sun in his apparent daily motion passes from east to west, slowly turns as a radius vector about the point O.

The extremity of this shadow travels in nearly a straight line across the board from E to D. This path, ED, is quite straight on March 21st and September 23rd, when the sun is on the equator, but curves slightly northwards when the sun is south of the equator, and

southwards when the sun is north of the equator. At any rate the circle GH if properly drawn will cut the line ED in two points, T in the forenoon and S in the afternoon. Then the line OM, passing through the mid-point of the arc TS, marks out the terrestrial meridian.

On account of the difficulty of accurately fixing the points T and S the result will be rendered more reliable by drawing several near concentric circles about O, and taking the mean of the observations upon all the circles.

The changing declination of the sun is a source of error, but the process is not sufficiently accurate to be much affected by this small error. Much greater errors will be introduced if the board is not accurately level, or the style OP is not accurately vertical. However, a rigidly vertical style is not necessary if the circles be drawn from P as a centre.

II. Equal altitudes of a star.

(*a*) *By azimuth.*

This method is a refinement upon the preceding one, and the instrument employed is the altazimuth.

We suppose that the field of view of the telescope is marked by two lines or threads, one being vertical and the other horizontal, as is usually the case, and that the instrument is in good adjustment with respect to collimation, level, &c. Choose a star, in the evening, at any distance from 20° to 60° east of the meridian, and direct the telescope so as to bring the star exactly upon the cross of the lines in the field of view. The instrument must now be clamped in altitude and must remain so throughout the observation. Let the azimuth circle be now read and let its reading be denoted by *a*. After

the star has passed the meridian it is to be followed by turning the instrument, in azimuth only, until the star comes to be again exactly on the cross of the lines. Let the reading of the azimuth circle taken now be denoted by b. Then if M denote the azimuth reading of the meridian,

$$M = \tfrac{1}{2}(a+b).$$

And by turning the instrument to this reading it is in the meridian, and unclamping and rotating in altitude the vertical line marks out the celestial meridian.

(*b*) *By time.*

The foregoing may be carried out with respect to time instead of azimuth, if the observer is supplied with a fairly good clock or watch. That the watch is correct as to time is not necessary.

Let t_1 be the ante-meridian time at which the star has the altitude a, and t_2 be the post-meridian time at which it has the same altitude. Then $\frac{1}{2}(t_1+t_2)$ is the time at which the star is on the meridian, and this is $\frac{1}{2}(t_1+t_2)-t_1$ or $\frac{1}{2}(t_2-t_1)$ after the first observation.

This result has no application on the first day of the observation. But let t' be the time at which the star has the same ante-meridian altitude on the following day. Then $t'+\frac{1}{2}(t_2-t_1)$ is the time at which the star will be on the meridian, and the instrument must be made to follow the star until this moment arrives, and is then to be clamped in azimuth with the star on the vertical thread. The instrument is then in the meridian, and motion in altitude will mark out the celestial meridian.

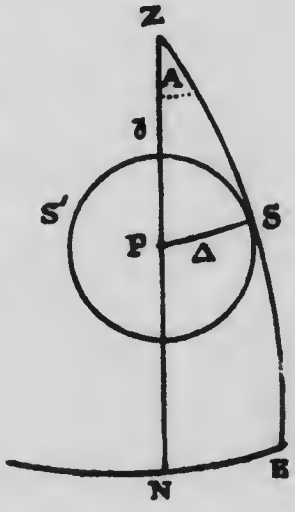
Ex. Let $t_1 = 8^{h.}\ 10^{m.}\ 22^{s.}$, $t_2 = 13^{h.}\ 40^{m.}\ 8^{s.}$, or $1^{h.}\ 40^{m.}\ 8^{s}$ by the watch. Then $\frac{1}{2}(t_2-t_1) = 2^{h.}\ 44^{m.}\ 53^{s.}$

On the following evening $t' = 8^h.\ 6^m.\ 26^s.$; so that the star will be on the meridian at $8^h.\ 6^m.\ 26^s. + 2^h.\ 44^m.\ 53^s.$, or $10^h.\ 51^m.\ 19^s.$.

With a good watch t' will be earlier than t_1 by about $3^m.\ 56^s.$ for every consecutive day.

III. Observation of a circumpolar star at greatest elongation.

A circumpolar star is a star not very far removed from the pole. Every star appears to make a daily revolution about the pole, the apparent path being a circle with the pole as centre. If the north polar distance of a star is less than the latitude of the place of observation, the star never sets, and if the north polar distance is less than the colatitude of the place the star crosses the meridian north of the zenith. At latitude north from 40° to 50°, several stars may be used as circumpolar stars. The principal ones in order of importance are 1. Polaris, 2. β. Ursae minoris, 3. λ Draconis, 4. α Ursae majoris, 5. α Cephei.



Z is the zenith, P the pole, ZN the north half of the meridian from the zenith to the horizon at N, S is a star whose apparent daily path is in the circle SS′ about P, and ZSH is the vertical circle which is tangent to the circle SS′ at S. $PS = \Delta$ is the north polar distance of the star and is given in the nautical almanac, and $PZ = \gamma$ is the colatitude of the place, and is supposed to be known.

By Napier's circular parts

$$\sin A = \frac{\sin \Delta}{\sin \gamma}$$

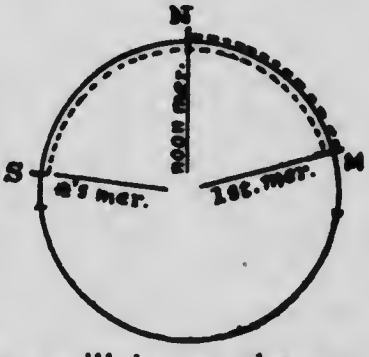
But, the observation being made with the alta-zimuth instrument, A is the azimuth of the instrument when moving in the vertical circle HZ. And if the instrument be turned in azimuth, towards the pole, through the angle A, it will be in the meridian.

When for any particular star the angle A becomes greatest, the star is said to be at its *greatest elongation*, and the motion of the star is then tangent to the vertical circle through it, as at S which marks its greatest eastern elongation, and at S′, its greatest western elongation.

In order to save watching and waiting it is desirable to know approximately when the star is at its greatest elongation.

For this purpose find the angle P, and change it into time, denoting its value by P_t.

Now let M denote the position of the first meridian on the equator, N denotes the position of the sun at noon, and S denotes the meridian of the star under consideration.



Then MN = S is the R.A. of the mean sun at noon, or the sidereal time at noon, and is to be taken from the nautical almanac for the day in question.

MS = α is the R.A. of the star, and NS = $\alpha - S$ is the distance of the star from the meridian expressed in time;—that is, after $\alpha - S$ hours the star will be on the meridian. Hence the times of greatest elongation are

$\alpha - S - P_t$ for eastern elongation,

and $\alpha - S + P_t$ for western elongation.

If α is less than S, we must add 24 hours to α, giving $\alpha + 24 - S$; and $\alpha - S + P_t$ must have 24 hrs. thrown out if it is greater than 24.

Ex. To find the azimuth at greatest elongation, and the times of greatest elongation of β Ursae minoris on May 20th, 1880.

From the nautical almanac for 1880 we have $\alpha = 14^{h.}\ 51^{m.}\ 9^{s.}$; $S = 3^{h.}\ 55^{m.}\ 3^{s.}$; $\Delta = 16^{\circ}21'20''$; and for Kingston $\gamma = 45^{\circ}46'40''$.

$$\text{Then } \sin A = \frac{\sin \Delta}{\sin \gamma} = 0{\cdot}39294,$$

and $A = 23^{\circ}8'15''$, the azimuth angle.

Again $\cos P = \tan \Delta \cot \gamma = 0{\cdot}28561.$

$\therefore$ $P = 73^{\circ}24'18''$, and $P_t = 4^{h.}\ 53^{m.}\ 37^{s.}$

Then $\alpha - S \pm P_t = 15^{h.}\ 49^{m.}\ 43^{s.}$ and $6^{h.}\ 2^{m.}\ 29^{s.}$

Hence the star will be at greatest elongation east at $6^{h.}\ 2^{m.}\ 29^{s.}$ in the evening, and west at $3^{h.}\ 49^{m.}\ 43^{s.}$ in the morning.

These times are more accurate than is necessary, as an approximation to within 5 minutes is quite sufficient.

There are several other methods of finding the meridian, but those given are the simplest and therefore the most practicable.

Another method is given in Art. 49.

Exercise.

1. For polaris on Sept. 6th, 1880, $\Delta = 88°40'18''$, $\alpha = 1^h.15^m.49^s.$, $S = 11^h.4^m.40^s.$, and $\gamma = 45°46'$ N., to find the azimuth at greatest elongation, and also the times of observation.

2. For α Ursae majoris on June 20th, 1880, $\Delta = 27°36'4''$, $\alpha = 10^h.56^m.21^s.$, $S = 5^h.56^m.17^s.$, $\gamma = 52°40'$ N., to find the azimuth at greatest elongation, and the times of observation.

45. *Problem. To find the latitude.*

The height of the north pole above the northern horizon is the latitude of the place of observation when the place is north of the equator; and similarly the height of the south pole gives the latitude of a place south of the equator.

But the pole is not marked by any object that can be observed, so that it is necessary to carry out our observations upon some of the heavenly bodies. The apparent altitude of any body not at the zenith is increased by refraction, so that the refraction for any altitude must be taken from a table of refraction, and as a correction refraction is always subtractive.

I. By the altitude of a circumpolar star.

If a star be not very far from the pole it may, in middle latitudes, be observed when on the meridian directly above the pole, or when on the meridian directly below the pole, *sub polo* as it is termed.

Let a be the observed altitude when above the pole, and r be the refraction due to this altitude, and Δ be the north polar distance of the star.

Then $\text{Lat.} = a - r - \Delta.$

And if a_1 be the observed altitude sub-polo, and r_1 be the refraction due to that altitude,

$$\text{Lat.} = a_1 - r_1 + \Delta.$$

The observation above the pole is to be preferred where convenient, since the refraction decreases as we rise towards the zenith.

Polaris is so near the pole that there is little choice between the observations.

Of course the altitude must be taken when the star is on the meridian.

II. By transit of a zenith star.

This method involves the same principle as I, except that a star is chosen which passes the meridian, or *culminates*, very near the zenith, and hence the refraction may be left out of account.

The zenith distance of the equator, measured along the meridian of the place, is the latitude.

Hence, if z be the zenith distance of the star at culmination, and δ be its declination,

$$\text{Lat.} = \delta \pm z,$$

according as the star crosses the meridian south or north of the zenith.

This is probably the most reliable of all methods, as it is free from the uncertainties of refraction, but it requires a special instrument known as a zenith sector, and especially adapted to make observations near the zenith.

III. By observing the sun's meridian altitude.

This method has the advantage of daylight observation; but it is more difficult to reduce than the others, because the sun has an appreciable disc, it is not fixed in declination as a star is, and it is subject to parallax.

Let δ = the sun's declination N.

δ' = the sun's change in declination for 1 hour, being positive when northwards.

P = the sun's parallax in altitude.

s = the sun's semidiameter.

r = the refraction.

λ = the longitude of the place of observation expressed in time and counted westward.

a = the observed altitude of the sun's upper limb.

Then $$\text{Lat.} = 90^\circ + s + r + \delta + \lambda\delta' - a - p.$$

With south declination δ is $-$, and δ' is $-$ when the sun is going southwards.

Ex. On March 3rd in long. $5^h.\ 5^m.\ 55^s.$ W. the sun's upper limb has an apparent altitude of $36^\circ 14'$ when on the meridian.

From the nautical almanac we have on March 3rd $\delta = -6^\circ 33' 13''$, $\delta' = 57''{\cdot}65$, $S = 16' 10''$, and $r = 1' 19''$; and from the parallax table $p = 8''$.

Whence

$$\text{Lat.} = 90^\circ + 16' 10'' + 1' 19'' - 6^\circ 33' 13'' + 4' 54'' - 36^\circ 14' - 8''$$
$$= 47^\circ 35' 2'' \text{ N.}$$

TIME.

46. Time has relation to the motions of the earth, that is, the apparent motions of the sun and stars. *Solar noon* is the time at which the sun's centre is on the local meridian, and the *solar day* is the time elapsing between two consecutive solar noons.

The time indicated by the sun is called solar time.

The angle between the plane of the local meridian and the plane of the meridian passing through the sun at any moment is the *hour angle* of the sun, and it expresses the angular distance of the sun from the local meridian.

Thus if the hour angle be 50°, the solar time is 8h. 40m. A.M. if the sun is east of the meridian, and 3h. 20m. P.M. if the sun is west. A sun dial keeps solar time.

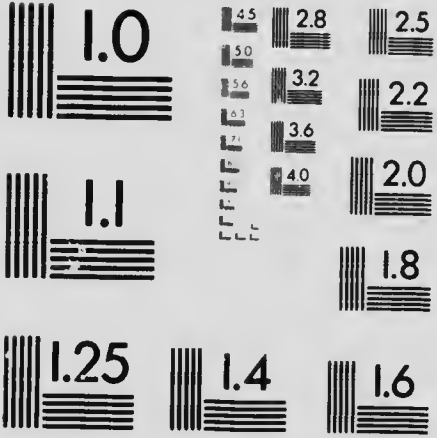
Owing to the obliquity of the ecliptic and the eccentricity of the earth's orbit, the solar days are not of equal length; or, in other words, solar time is not measured out uniformly.

To overcome this difficulty, the astronomer supposes a fictitious sun called the *mean* sun which moves in the equator with perfect uniformity, and which, on the average, completes the circuit of the year in the same time as the real sun.

The time indicated by this mean sun is called mean time, and is kept by every good and well regulated mean time clock. Mean noon, or as it is commonly called 12 o'clock, is the time at which the centre of the mean sun is on the meridian, and the time elapsing between two consecutive mean noons is a mean day; and all mean days are equal.

MICROCOPY RESOLUTION TEST CHART

(ANSI and ISO TEST CHART No. 2)



APPLIED IMAGE Inc

1653 East Main Street
Rochester, New York 14609 USA
(716) 482 - 0300 - Phone
(716) 288 - 5989 - Fax

The difference between solar time and mean time is known as the *equation of time*. This is given in the nautical almanac for every day throughout the year.

Now, the mean sun being only a fiction, observations must be made upon the real sun; and the results of such observations are then reduced to mean time by means of the equation of time.

47. To find the solar time, and thence the mean time.

I. By transit of the sun across the meridian.

This requires an instrument, such as the altazimuth, set so that the vertical thread moves exactly in the meridian. At the sun's culmination, let t_1 denote the time at which the western limb of the sun touches the vertical thread, and t_2 denote that at which the eastern limb touches the thread.

Then at $\frac{1}{2}(t_1+t_2)$ the sun's centre was on the meridian. This was solar noon, and the amount by which $\frac{1}{2}(t_1+t_2)$ exceeds or falls short of 12 hours is the error of the time-piece from solar time, the time-piece being fast on solar time when $\frac{1}{2}(t_1+t_2) > 12$, and slow when < 12.

Ex. The watch gives $11^{h.}\,58^{m.}\,10^{s.}$ at first contact, and $12^{h.}\,0^{m.}\,26^{s.}$ at last contact; $\frac{1}{2}(t_1+t_2)=11^{h.}\,59^{m.}\,18^{s.}$; and the watch is $12^{h.}-11^{h.}\,59^{m.}\,18^{s.}=42^{s.}$ slow on solar time.

This observation being made on July 13th the equation of time is given as $5^{m.}\,31^{s.}$ to be added to apparent time; or, mean time is $5^{m.}\,31^{s.}$ ahead of solar time. Hence the watch is $5^{m.}\,31^{s.}+42^{s.}=6^{m.}\,13^{s.}$ slow on mean time.

Ex. On Dec. 1st a watch showed $11^{h.}\ 50^{m.}\ 13^{s.}$ and $11^{h.}\ 52^{m.}\ 33^{s.}$ at the two contacts with the vertical thread. The equation of time gives solar time $10^{m.}\ 35^{s.}$ ahead of mean time. Find the error of the watch on mean time.

II. By altitude of the sun out of the meridian.

Let a be the observed altitude of the sun's upper limb taken 2 or 3 hours before or after noon.

Then the true altitude of the sun's centre is $a+p-s-r$, where p, s, r are respectively the parallax in altitude, the semidiameter and the refraction. The complement of this result is z, the zenith distance of the sun's centre.

From the nautical almanac we take the sun's declination for the time of observation, and the complement of this is Δ, the sun's N.P.D.

And we suppose the colatitude, γ, of the place to be known.

Then we have the spheric triangle whose vertices are P, Z, S, and whose opposite sides are z, Δ, γ, and we have to find the angle at P, this being the hour angle, h.

We may take either of the forms—

$$\cos h = \frac{\cos z - \cos\gamma \cos\Delta}{\sin\gamma \sin\Delta},$$

or
$$\cos\frac{h}{2} = \sqrt{\left\{\frac{\sin s \sin(s-z)}{\sin\gamma \sin\Delta}\right\}}.$$

Change h to time and call it H. Then H gives the number of hours that the sun is distant from the meridian.

Now if t be the time of the observation as shown by the watch.

$$\left.\begin{array}{ll} t+H-12 & \text{for A.M. observation} \\ t-H & \text{for P.M. observation} \end{array}\right\} = E.$$

And E is the error of the watch on solar time, the watch being fast or slow according as E is positive or negative.

Ex. On May 10th, 1900, in Lat. 44°13′ N. and Long. $5^{h.}\,6^{m.}$ west, at $8^{h.}\,9^{m.}\,44^{s.}$, as shown by the watch, the apparent altitude of the sun's upper limb was 36°10′. To find the error of the watch on mean time.

We take from the Nautical Almanac for 1900, at 8h. 9m. A.M. at the given longitude—

sun's declination $= 17°35′17″$, $\therefore \quad \Delta = 72°24′43″$.

Equation of time $= 3^{m.}\,46^{s.}$, solar time being fast.

$s = 15′52″$, $r = 1′20″$, $p = 8″$; and hence $a+p-s-r$ $= 35°52′56″$, and hence $z = 54°7′4″$.

Also $\gamma = 45°47′$.

Then $$\frac{\cos z - \cos\gamma \cos\Delta}{\sin\gamma \sin\Delta} = \cdot 54943,$$

and $h = 56°40′$, and $H = 3^{h.}\,46^{m.}\,40^{s.}$

$$\therefore \quad t+H-12^{h.} = 3^{m.}\,36^{s.}$$

Thus the watch is $3^{m.}\,36^{s.}$ slow on solar time, and 10^{s} fast on mean time.

III. By equal altitudes of the sun.

Let the sun's upper limb be observed at the same altitude, before noon and again after noon, and let t_1 and

t_2 be the times shown by the watch at the moments of observation.

Then if the observations be made about June 23rd or Dec. 21st, the error of the watch on solar time will be

$$\tfrac{1}{2}(t_1 + t_2 - 12).$$

If, however, the observations be made not near the foregoing dates, the change of declination of the sun between the two observations must be taken into account.

In the triangle whose sides are z, Δ, γ, we have

$$\cos z = \cos\gamma \cos\Delta + \sin\gamma \sin\Delta \cos h.$$

The only variables in this expression are Δ which varies with the sun's declination, and h which depends upon Δ for its variation. Hence differentiating—

$$0 = (-\cos\gamma \sin\Delta + \sin\gamma \cos\Delta \cos h)\,d\Delta - \sin\gamma \sin\Delta \sin h \cdot dh)$$

Whence
$$dh = \left(\frac{\tan\delta}{\tan h} - \frac{\tan l}{\sin h}\right) d\Delta$$

or
$$dh = \left(\frac{\tan l}{\sin h} - \frac{\tan\delta}{\tan h}\right) d\delta.$$

Now $d\delta$ may be taken to represent the change in the sun's declination for 1 hour, and is given in seconds of arc in the almanac.

Hence $\left(\frac{\tan l}{\sin h} - \frac{\tan\delta}{\tan h}\right) \frac{d\delta}{15}$ is the change in the hour angle for 1 hour. Denote this by h'.

The time elapsing between the observations is $t_2 + 12 - t_1$, and $h'(t_2 + 12 - t_1)$ is the whole change in hour angle, $= H$ say.

This quantity must be subtracted from t_2 when the sun is going northwards and $d\delta$ is $+$, and must be added to t_2 when the sun is going southwards and $d\delta$ is $-$.

$$\therefore \quad \tfrac{1}{2}(t_1 + t_2 - H - 12)$$

is the error of the watch on solar time.

Ex. On March 5th, 1856, at lat. 38°59′ N. the sun was observed at equal altitudes at $8^{h}\cdot 12^{m}\cdot 20^{s}\cdot$ A.M., and $3^{h}\cdot 49^{m}\cdot 35^{s}\cdot$ P.M., to find the error of the watch on mean time.

$t_1 = 8^{h}\cdot 12^{m}\cdot 20^{s}\cdot$, $t_2 = 3^{h}\cdot 49^{m}\cdot 35^{s}\cdot$, $l = 38°59′$ N.; and from the almanac, $\delta = -5°46′23″$, $d\delta = +58″\cdot 1$.

Equation of time, sun slow $11^{m}\cdot 35^{s}\cdot$

Take

$$h = 12^{h}\cdot - 8^{h}\cdot 12^{m}\cdot 20^{s}\cdot = 3^{h}\cdot 47^{m}\cdot 40^{s}\cdot = 56°55′,$$

Then $$\left(\frac{\tan 38°59′}{\sin 56°55′} + \frac{\tan 5°46′23″}{\tan 56°55′}\right)\frac{58\cdot 1}{15} = h' = 3^{s}\cdot 98.$$

The elapsed time, $(t_2 + 12 - t_1) = 7\cdot 62$ hours.

And $3^{s}\cdot 98 \times 7\cdot 63 = 30^{s}\cdot 33 = H$.

And $\tfrac{1}{2}(t_1 + t_2 - H - 12) = 0^{m}\cdot 42^{s}\cdot$ to the nearest second, and the watch is fast $42^{s}\cdot$ on solar time.

But mean time is fast $11^{m}\cdot 36^{s}\cdot$ on solar time.

$\therefore$ The watch is slow $10^{m}\cdot 54^{s}\cdot$ on mean time.

48. Sidereal time.

Sidereal time is measured out by the apparent diurnal revolution of the celestial sphere. *The first point of Aries*—called also the ascending node, and the vernal

equinox—is the starting point, and when this point is on the local meridian the sidereal time is $0^{h}\cdot 0^{m}\cdot 0^{s}\cdot$ Thence it is counted around through 24 hours. If a sidereal clock, that is, a clock keeping sidereal time, is placed with its dial looking northwards the hour hand will point perpetually to the first meridian. And thus the sidereal time is the same thing as the right ascension of the local meridian.

The right ascension of the local meridian at mean noon is the right ascension of the mean sun at that time and is given in the almanac for every day in the year under the column headed "sidereal time".

If α be the right ascension of a star, the star will be on the local meridian when the right ascension of the meridian is α, i.e., when the sidereal clock points to the time α.

The almanac gives also the mean time of transit of the first point of Aries, i.e., the mean time at which the sidereal clock shows $0^{h}\cdot 0^{m}\cdot 0^{s}\cdot$ for every day in the year.

Thus on Aug. 1st, 1880, we find $15^{h}\cdot 15^{m}\cdot 36^{s}\cdot\cdot 65$ as the mean time at which the sidereal clock shows $0^{h}\cdot 0^{m}\cdot 0^{s}\cdot$, or the sidereal day begins.

The tropical or equinoctial year contains 365·2422 mean days and 366·2422 sidereal days. So that

$$\frac{366\cdot 2422}{365\cdot 2422} = 1\cdot 002738$$

is a multiplier which changes an interval given in mean time, to the same interval in sidereal time.

Thus 24 days in mean time $= 24 \times 1\cdot 002738$ days in sidereal time, $= 24^{d}\cdot 1^{h}\cdot 34^{m}\cdot 37^{s}\cdot$

Similarly $\frac{365\cdot2422}{366\cdot2422}=0\cdot997269$

is the multiplier which changes a sidereal interval into a mean time interval.

If, at any date, T_m denote the mean time, and D be the days and decimals of a day elapsing since the vernal equinox, and T_s be the corresponding sidereal time,

$$T_s=T_m+0\cdot002738D.$$

Ex. If a star has its R.A. $15^h\ 20^m\cdot10^s$, to find when it will be upon the meridian on Aug. 2nd.

We will assume that the vernal equinox was on March 21st at noon. Then from March 21st to Aug. 2nd at noon is 134 days. And $134\times0\cdot002738=\cdot3669$ days$=8^h\cdot48^m\cdot20^s$.

And $15^h\cdot20^m\cdot10^s-8^h\ 48^m\cdot20^s=6^h\cdot31^m\cdot50^s$; and the star will be on the meridian at $6^h\cdot31^m\cdot50^s$ in the evening.

Owing to our assumption of the time of the vernal equinox being only approximately true, this is only an approximation. But with the assumption here made the calculated time will never be 4 minutes out.

EXERCISE.

1. At $8^h\cdot20^m\cdot14^s$ a.m. by a watch $1^m\cdot20^s$ slow on solar time, what is the sun's hour angle?

2. If in Ex. 1 the equation of time be, sun fast $6^m\cdot15^s$, find the hour angle of the mean sun.

3. If the semi-diameter of the sun is 16′3″, and we observe the 2nd limb on the meridian at $1^m\cdot4^s$ past 12 noon, find the error of the watch on solar time.

4. The apparent altitude of the sun's upper limb is observed to be $21^\circ 36'$ at lat. $44^\circ 13'$ north. If $s = 16'\,9''$, $r = 1'\,10''$, $p = 8''$, and $\delta = 6^\circ 10' 6''$ S., and the watch reads $8^h\cdot 18^m$, find the error on solar time.

   And if the sun be $11^m\cdot 46^s\cdot 8$ slow, find the error on mean time.

5. The sun is observed at equal altitudes at $9^h\cdot 10^m\cdot 16^s$ a.m., and $2^h\cdot 40^m\cdot 50^s$ p.m. Find the error of the watch on solar time, if $\delta = 4^\circ 48' 12''$ N., $d\delta = 57''\cdot 7$, $l = 45^\circ 40'$ N.

6. The R.A. of Altair is $19^h\cdot 45^m$. Find about when Altair will be on the meridian on Sept. 5th.

7. If t be the time corresponding to $\sin^{-1}(\tan\delta \tan l)$ where δ is the sun's declination and l is the latitude, the length of the day is $2(6+t)$ hours; and the sun rises $\sin^{-1}(\sin\delta \sec l)$ degrees north of east if δ is positive, and south of east if δ is negative.

49. *To find the meridian from a single altitude of the sun.*

At some convenient time, as 2 or 3 hours before noon, observe the sun's altitude, and let t be the time of observation as shown by the watch.

Let h be the hour angle as determined from the observation. At the time $t+h$, by the watch, the sun's centre will be approximately on the meridian.

At this moment bring the vertical thread to touch the sun's limb. Then

$$s \operatorname{cosec}(l-\delta)$$

will be the azimuth of the instrument, where s is the

sun's semidiameter, l the latitude, and δ the sun's declination, positive when north and negative when south.

For greater accuracy the following correction for the variable motion of the sun is necessary.

Let v be the variation in the equation of time for one hour, as taken from the almanac; then

with solar time fast; equation increasing } subtract
or, solar " slow; " decreasing } tv from t.

With solar time fast; equation decreasing } add
or " " " slow; " increasing } tv to t.

Ex. On Jan. 20th, 1880, at latitude 44°30′ N., and at $10^{h}\cdot10^{m}$ a.m. the altitude of the sun is observed, and the resulting hour angle is worked out to be $1^{h}\cdot48^{m}\cdot12^{s}= 1^{h}\cdot803$. Also $v=0^{s}\cdot74$, the sun is slow, and the equation of time is increasing.

$\therefore \quad tv=1\cdot803\times0\cdot74=2^{s}\cdot0$ to be subtracted from t, leaving $t-tv+h=11^{h}\cdot58^{m}\cdot10^{s}$ as the time when the sun's centre will be on the meridian.

Then $\delta=-20°11′$, and $s=16′\cdot29$, so that

$$16′\cdot29\times\text{cosec.}\,(64°41′)=18′\cdot02,$$

which is the azimuth angle of the instrument.

List of Errata.

Page 10, Ex. 6, for $\pi - 3\cos^{-1}\frac{1}{3}$, read $3\cos^{-1}\frac{1}{3} - \pi$.
" 28, 4th line from bottom, for $-$ read $+$.
" 73, 1st line, for P_1 read P_t.
" 74, Ex. 1, for $\triangle$ read δ.
" 82, 18th line, for 7·63 read 7·62.
" 82, lines 5 and 18, for $\frac{1}{2}(l_1 + l_2 - H - 12)$ read $\frac{1}{2}(l_1 + l_2 - H) - 12$.

Answers to Exercises.

These answers when given in angle are to the nearest 10", except in a few cases where greater accuracy is required.

Page 9.

1. 59°31′40″. 2. 132°1′. 4. 16°20′6″.
8. 69·115..mi. 9. 281 m. nearly.

Page 12.

1. 26°4′50″. 2. Angle = 90°28′20″; dist. = 507·8 m.
3. 47°27′40″, 54°36′30″, 63°38′50″.

Page 16.

1. 1587·1 5. $\cos^{-1}\frac{1}{3}$ or 70°31′44″.
6. 87°15′, 113°11′10″, 118°33′10″.
9. $\sin a = \frac{2}{\sin B \sin C}\sqrt{\{-\cos S \cos(S-A)\cos(S-B)\cos(S-C)\}}$

Page 19.

1. $C = 42^\circ 24' 20''$, $a = 119^\circ 58' 30''$, $b = 103^\circ 26' 30''$.
2. $63^\circ 9' 40''$, $47^\circ 32' 40''$.
3. From A to B, $17^\circ 31' 50''$ west of north.
   From B to A, $26^\circ 26' 50''$ east of south.
   Distance from A to $B = 1705{\cdot}6$ miles.
4. $45^\circ 13' 40''$.

Page 28.

2. $17^\circ 13' 20''$.
3. $\cos^{-1}\frac{1}{3}$, or $70^\circ 31' 44''$.
4. $\frac{\pi}{2}$ or 90°.

Page 31.

1. $B = 62^\circ 36' 20''$, $C = 145^\circ 42' 30''$, $a = 31^\circ 43\ 30''$.
2. $A = 52^\circ 15' 30''$, $b = 98^\circ 1'$, $c = 90^\circ 7' 50''$.

Page 34.

1. 995·6 m.
2. Lat. $= 38^\circ 40' 40''$ N. or $53^\circ 40' 20''$.
3. Find A by the sine formula, and then find c by Napier's analogy 3 or 4.

Page 37.

4. 582·7 m.

Page 39.

2. 2,736,955 sq. miles.
3. $81''{\cdot}88$., 6225 sq. miles.
4. Area = 6224·3 sq. m. E thus found $= 81''{\cdot}87$.
5. $5^\circ 8' 30''$.

Page 44.

1. $24^\circ 11' 30''$.
2. $2' 36''{\cdot}3$.
3. $129''\ 1$.
4. 552·86 miles.
5. 368·5 m.

Page 46.

1. N.E. 35′19″ E., or N. 45°35′19″ E.
2. E. 0°33′54″·6 S.
3. Lat. = 53°2′34″ N., long. 27°38′41″ W.

Page 47.

1. 251 m. nearly. 2. $4^h 38^m 30^s$, or $5^h 33^m 20^s$.
3. 9331 miles = the distance along a great circle and therefore the shortest distance.
4. Lat. = 29°5′7 N., long. = 86°20′ W.

Page 51.

1. 20°18′ N. 2. γ = E. 11°44′ N.
5. Great circle distance = 2128 m.; loxodrome distance = 2132·6 m. γ for great circle = 35°44′, and for loxodrome = 41°49′.
6. 1367·6 m.

Page 58.

1. Latitude, 32°0′0″; 32°12′31″; 32°25′2″; 32°37′33″; 32°50′3″; 33°2′33″.
   Angle, 16°0′0″; 16°2′15″; 16°4′32″; 16°6′50″; 16°9′10″; 16°11′31″.
2. Course = N.47°12′40″ W. Distance = 398·5 m.
3. The area of a district bounded by two meridians and two parallels of latitude is given by the formula $r^2\theta(\sin l' - \sin l)$, or $2r^2\theta \sin \frac{1}{2}(l'-l) \cos\frac{1}{2}(l'+l)$, where r is the earth's radius, θ is the radian measure of the angle between the meridians, and l and l' are the lower and the higher latitudes.
   Diff. = 0·38 miles. Area = 1300·2 sq. miles.

4. Deflection = 37′·6. Departure 162 feet nearly.
5. Lat. 29°24′20″ N. Course N. 54°0′10″ W. Long. 33°46′30″ W.
6. 1′53″ nearly.

PAGE 63.

1. 8°5′40″. 2. 11′10″·5. 3. 44·33 m.
6. 32 miles nearly. 7·21 inches.

PAGE 74.

1. Azimuth = 1°51′15″; times $8^h16^m21^s$ p.m., $8^h5^m57^s$ a.m.
2. Azimuth = 35°38′25″; times $0^h34^m10^s$ p.m., $9^h25^m50^s$ p.m.

PAGE 84.

1. $3^h28^m36^s$. 2. $3^h44^m41^s$. 3. Slow $0^s·2$.
4. Watch slow on solar time 13^m53^s; slow on mean time $25^m39^s·8$.
5. Watch slow 4^m42^s. 6. About 8^h40^m p.m.